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DESIGNING ENGINEERING TESTS BY USE OF SAMPLE INFORMATION

THE FIRST PRINCIPLE OF SAMPLE TESTING

The Measurement of Evidence:

We ask "Evidence of what ?" .

ANSWER: Evidence in favor of some hypothesis.

What is meant by an hypothesis ?

ANSWER: An hypothesis is a claim (i.e., a contention or declaration) that is made regarding the behaviour of a statistical variable under consideration.

EXAMPLES OF HYPOTHESES

1. The statement "This product has an average service life of 1000 hours." is an hypothesis.

2. The declaration "Design B has a median life which is at least 25% better than the median life of Design A under like operating conditions." is an hypothesis.

3. The contention "We claim a population B_{10} Life for our product which is at least 95% of the B_{10} Life shown by our test sample." is an hypothesis.

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MATHEMATICAL DEFINITION OF THE EVIDENCE IN FAVOR OF AN HYPOTHESIS

Let C = CONFIDENCE (i.e., probability that the hypothesis under consideration will turn out to be true.)

Then, (1 - C) = COMPLEMENTARY CONFIDENCE (i.e., probability that the hypothesis under consideration will turn out to be false.)

$$\frac{C}{1 - C} = \text{ODDS RATIO in favor of the hypothesis}$$

$$\ln\left(\frac{C}{1 - C}\right) = \text{EVIDENCE in favor of the hypothesis}$$

DEFINITION

The EVIDENCE in favor of an hypothesis is the natural logarithm of the odds ratio in favor of the hypothesis.

(See Volume 1 , Bulletin 4)

EVIDENCE VERSUS CONFIDENCE INDEX

TABLE 1

C CONFIDENCE	E UNITS OF EVIDENCE
50 %	0
60 %	0.4055
70 %	0.8473
80 %	1.3863
90 %	2.1972
95 %	2.9444
99 %	4.5951
99.9 %	6.9068
99.99 %	9.2102

DERIVATION OF THE INVERSE FORMULA

$$E = \ln\left(\frac{C}{1 - C}\right)$$

$$\frac{C}{1 - C} = e^{E}$$

$$C = \frac{1}{1 + e^{-E}}$$

CONFIDENCE INDEX VS. WHOLE UNITS OF EVIDENCE

TABLE 2

E UNITS OF EVIDENCE	C CONFIDENCE
0	50 %
1	73.11 %
2	88.08 %
3	95.26 %
4	98.20 %
5	99.33 %
6	99.75 %
7	99.91 %
8	99.97 %
9	99.988 %
10	99.995 %

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CALCULATING EVIDENCE FROM A SAMPLE

RULE 1 : Whenever confidence is obtained from a Z-SCORE , the EVIDENCE is this Z-score multiplied by $\frac{\pi}{\sqrt{3}}$.

EXAMPLE 1

For the hypothesis : B_Q Life $\geq$ X_o .

Based on N observations, the CONFIDENCE C is obtained from the Z-score

$$Z_C = b\sqrt{Q\,N}\ \ln\left(\frac{B_Q}{X_o}\right)$$

(b = Weibull Slope)
($Q \leq .5$)

According to the LOGISTIC FORMULA:

$$C = \frac{1}{1 + \mathrm{EXP}(-\frac{\pi}{\sqrt{3}} Z_C)} = \frac{1}{1 + \mathrm{EXP}(-1.8138\, Z_C)}$$

$$E = 1.8138\ b\sqrt{Q\,N}\ \ln\left(\frac{B_Q}{X_o}\right)$$

or

$$E = 1.8138\ Z_C$$

CONCLUSION

Evidence is proportional to the SQUARE ROOT of the SAMPLE SIZE .

RULE 2 : Whenever confidence is obtained from a t-SCORE TO COINCIDENCE , the EVIDENCE is this t-score multiplied by $\frac{\pi K}{\sqrt{3}}$, where

$$K = \sqrt{1 + \frac{\sqrt{N_1 N_2}}{\frac{1}{2}(N_1 + N_2)}}$$

(N_1 = First Sample Size, and N_2 = Second Sample Size .)

NOTE: For two EQUAL SAMPLE SIZES (i.,e., $N_2 = N_1$) the value of K is $\sqrt{2}$.)

EXAMPLE 2

For two equal samples of size N each (yielding a TOTAL DEGREES OF FREEDOM $T = (N-1)^2$) at QUANTILE $Q \leq .5$, we have the CONFIDENCE FORMULA (See VOLUME 5, BULLETIN 2) :

$$C = \frac{1}{1 + \rho^{-b\sqrt{2NQ}}}$$

(b=Weibull Slope)

($\rho = B_Q$ Life Ratio)

$$\frac{C}{1 - C} = \rho^{+b\sqrt{2NQ}}$$

$$E = \ln\left(\frac{C}{1 - C}\right) = b\sqrt{2NQ}\,\ln\rho = \frac{\pi\sqrt{2}}{\sqrt{3}}\,\hat{t}_C = 2.565\,\hat{t}_C$$

$$\hat{t}_C = \frac{b}{2.565}\sqrt{2NQ}\,\ln\rho = .55\,b\sqrt{NQ}\,\ln\rho$$

CONCLUSION: EVIDENCE is proportional to the SQUARE ROOT of the SAMPLE SIZE.

EXAMPLE 3

The confidence that a population B_Q Life as estimated from a sample of size N is at least the fraction (1 - P) of the sample B_Q is obtained from the Z-score

$$Z_C = \frac{B_Q - (1 - P) B_Q}{\sigma_{B_Q}} = \frac{P B_Q}{\sigma_{B_Q}}$$

Now $$\sigma_{B_Q} = \frac{1}{f_Q} \sqrt{\frac{Q(1 - Q)}{N}}$$

where f_Q = PDF ordinate at quantile Q .

$$Z_C = \frac{P B_Q f_Q \sqrt{N}}{\sqrt{Q(1 - Q)}}$$

$$\text{EVIDENCE} = E = \frac{\pi}{\sqrt{3}} Z_C = \frac{\pi P B_Q f_Q \sqrt{N}}{\sqrt{3 Q(1 - Q)}}$$

CONCLUSION

Evidence is proportional to the SQUARE ROOT of the SAMPLE SIZE .

THE SECOND PRINCIPLE OF SAMPLE TESTING

SAMPLE EFFECTIVENESS PRINCIPLE: The EVIDENCE PER ITEM TESTED decreases with sample size, even though the TOTAL ACCUMULATED EVIDENCE increases.

Since evidence is directly proportional to the square root of the sample size, it follows that the MULTIPLYING FACTOR for the INCREASE IN EVIDENCE in going from a sample of size N to a sample of size N + 1 is

$$W = \frac{E_{N+1}}{E_N} = \frac{\sqrt{N+1}}{\sqrt{N}} = (1 + 1/N)^{\frac{1}{2}}$$

For values of N from N=1 to N = 100, this factor W is tabulated below :

TABLE 3

N	W
1	1.414
2	1.225
3	1.155
4	1.118
5	1.095
6	1.080
7	1.069
8	1.061
9	1.054
10	1.049
20	1.025
30	1.017
40	1.012

TABLE 3 ----Concluded

N	W
50	1.010
60	1.008
70	1.007
80	1.006
90	1.0055
100	1.0050

The EVIDENCE PER ITEM TESTED is $V_N = \frac{E_N}{N} = \frac{\lambda \sqrt{N}}{N}$

or, $V_N = \frac{\lambda}{\sqrt{N}} = \frac{E_1}{\sqrt{N}}$

Thus, the EVIDENCE PER ITEM TESTED is a decreasing function defined as the EVIDENCE FROM THE 1st ITEM TESTED divided by THE SQUARE ROOT OF THE TOTAL NUMBER TESTED. The following table shows this behaviour numerically for sample sizes from $N = 1$ to $N = 128$.

TABLE 4

N	EVIDENCE PER ITEM TESTED
1	E_1
2	.7071 E_1
4	.5000 E_1
8	.3536 E_1
16	.2500 E_1
32	.1768 E_1
64	.1250 E_1
128	.0884 E_1

CONCLUSION: Evidence per item tested decreases at the same rate as $\frac{1}{\sqrt{N}}$.

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THE THIRD PRINCIPLE OF SAMPLE TESTING

THE CONFIDENCE SELECTION RULE: The appropriate confidence level C for any situation is such that

$$C \text{ (Dollar Gain From A Correct Decision)} = (1 - C) \text{ (Dollar Loss From A Wrong Decision)}$$

$$\therefore \quad \frac{C}{1 - C} = \frac{\text{Dollar Loss From A Wrong Decision}}{\text{Dollar Gain From A Correct Decision}}$$

$$\text{Evidence} = E = \ln\left(\frac{C}{1 - C}\right) = \ln\left(\frac{\text{Dollar Loss From A Wrong Decision}}{\text{Dollar Gain From A Correct Decision}}\right)$$

CONCLUSION

The optimum number of UNITS OF EVIDENCE for a hypothesis test is given by the natural logarithm of the ratio of the dollars lost if the hypothesis turns out to be false to the dollars gained if the hypothesis turns out to be true .

Using this principle, we construct TABLE 5 on the next page for the OPTIMUM UNITS OF EVIDENCE corresponding to different DOLLAR RATIOS for LOSSES vs. GAINS.

TABLE 5

(Dollar Loss For False Hypothesis / Dollar Gain For True Hypothesis)	OPTIMUM CONFIDENCE	OPTIMUM UNITS OF EVIDENCE
1.0	50 %	0
1.2	54.55 %	.1823
1.4	58.33 %	.3365
1.6	61.54 %	.4700
1.8	64.29 %	.5878
2.0	66.67 %	.6931
2.5	71.34 %	.9163
3.0	75 %	1.0986
4.0	80 %	1.3863
5.0	83.33 %	1.6094
10.0	90.91 %	2.3026
50	98.04 %	3.9120
100	99.01 %	4.6052
500	99.80 %	6.2146
1000	99.90 %	6.9078

THE FOURTH PRINCIPLE OF SAMPLE TESTING (SEQUENTIAL TESTING)

FACT OF LIFE : On a new product the statistical parameters of life are unknown.

CONSEQUENCES OF THIS FACT OF LIFE:

Test sample size cannot be determined in advance.

WHAT IS THE WAY OUT OF THIS DIFFICULTY ?

Use STEP by STEP testing, i.e., SEQUENTIAL TESTING , where we test one item at a time and stop testing when SUFFICIENT EVIDENCE has been accumulated to yield the proper optimum confidence corresponding to the DOLLAR RATIO of LOSSES versus GAINS for FALSE and TRUE outcomes, respectively, in the case of the hypothesis under consideration .