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PREDICTING FATIGUE LIFE FROM ULTRASONIC INDICATIONS

Making a daily effort to apply tried and tested concepts is the best way to help build progress assurance. This operating principle applies in all phase of man's activities. Certainly it is the case in scientific research.

The problem of predicting fatigue life is by universal admission a statistical or probabilistic one. A tested and tried probability theory in fatigue is the so-called Weibull Theory. Therefore, in studying non-destructive measuring techniques (such as ultrasonics) and how they correlate with fatigue life it is advisable to employ the Weibull theory as a research program guide, instead of going out on some wild uncharted seas.

It is already decades ago since Weibull came out with a theory which stated that

$$\text{Life Varies As} \quad \frac{1}{\left[\text{Stressed Volume}\right]^{\frac{1}{\text{Weibull Slope}}}}$$

This Weibull Law for <u>SIZE EFFECT</u> in fatigue will serve as our basis for studying how <u>ULTRASONIC INDICATIONS</u> are related to <u>FATIGUE LIFE</u>.

The Weibull Law which is based on straightforward logic and the probability of joint but independent events certainly deserves a fair trial. This is our intention in the present investigation.

STRESSED VOLUME THEORY

If a unit of volume is subjected to stress and has a probability $P_1(x)$ of surviving for time x, then two units of volume under the same stress have a probability

$P_2(x) = \left[P_1(x)\right]^2$ of both surving for time x.

Similarly, three units of volume subjected to this same stress have a probability

$P_3(x) = \left[P_1(x)\right]^3$ of all surving for time x.

In general, V units of volume will have a probability $P_V(x) = \left[P_1(x)\right]^V$ of surviving for time x at the same original stress. Unless all V units of volume making up a specimen survive the time x the specimen is a failure in time x. $P_1(x)$ is known as the survivorship function for a unit volume. If a unit volume has a Weibull survivorship function, then

$$P_1(x) = e^{-(x/\theta_1)^b} \qquad (1)$$

where θ_1 = characteristic life of the unit volume

b = Weibull slope

It follows, therefore, that V units of volume have a joint survivorship function

$$P_V(x) = \left[P_1(x)\right]^V = e^{-V(x/\theta_1)^b} \qquad (2)$$

or, $$P_V(x) = e^{-\left(\frac{x}{\theta_1/V^{1/b}}\right)^b} = e^{-\left(\frac{x}{\theta_V}\right)^b} \qquad (3)$$

where θ_V = characteristic life of V units of volume (jointly)

It can be seen that $$\theta_V = \frac{\theta_1}{V^{1/b}} \qquad (4)$$

In other words, <u>LIFE OF V units</u> of volume = $\dfrac{\text{constant}}{V^{1/b}}$ (5)

This last relation is known as the **SIZE EFFECT THEOREM** for materials having Weibull life distributions. This is the basic equation for STRESSED VOLUME THEORY when failures obey a Weibull law of distribution.

MATERIAL DEFECTS AND THEIR RELATION TO FAILURE

Materials under stress fail because flaws or discontinuities allow fracture to initiate and propagate. The more flaws present in a material of given volume the higher the probability of failure in a given time becomes. This is nothing other than a version of the SIZE EFFECT THEOREM. Why? Because each flaw represents a stressed volume, and the more flaws present the greater the stressed volume. This logically follows, especially since flaws produce stress concentration.

Furthermore, large defects constitute large stressed volumes and small defects constitute small stressed volumes.

The TOTAL STRESSED VOLUME due to defects is then proportional to two things:

FIRST: The number of defects

SECOND: The volume per defect

Suppose a specimen has N defects, each of volume V_1.

The TOTAL DEFECT VOLUME is then $V = NV_1$ (6)

According to the SIZE EFFECT THEOREM

$$\text{LIFE} = \frac{\text{constant}}{V^{1/b}} \tag{7}$$

Putting $V = NV_1$ in (7) we obtain the formula

$$\text{LIFE} = \frac{\text{constant}}{N^{1/b} V_1^{\ 1/b}} \qquad (8)$$

If we think of each defect as a sphere of diameter D_1 , it follows that

$$V_1 = 1/6\pi D_1^{\ 3} \qquad (9)$$

(Note: Even if defects are not spherical, it still follows that volumes of similar defects are proportional to the cubes of corresponding dimensions.)

Thus, we may state that

$$\text{LIFE} = \frac{\text{constant}}{N^{1/b} \ D_1^{\ 3/b}} \qquad (10)$$

This last equation is the theoretical prediction equation for life in terms of the total number of defects N, and the diameter per defect D_1 . An important theoretical fact to be noted is that the exponent of the defect diameter is 3 times the exponent on the number of defects.

ULTRASONIC INDICATIONS AND THEIR SIGNIFICANCE

An ultrasonic trace of a specimen (say a bearing race) yields a series of peaks or bumps (frequently called "Blips") of varying height and spacing typified by what is sketched in FIGURE 1 below:

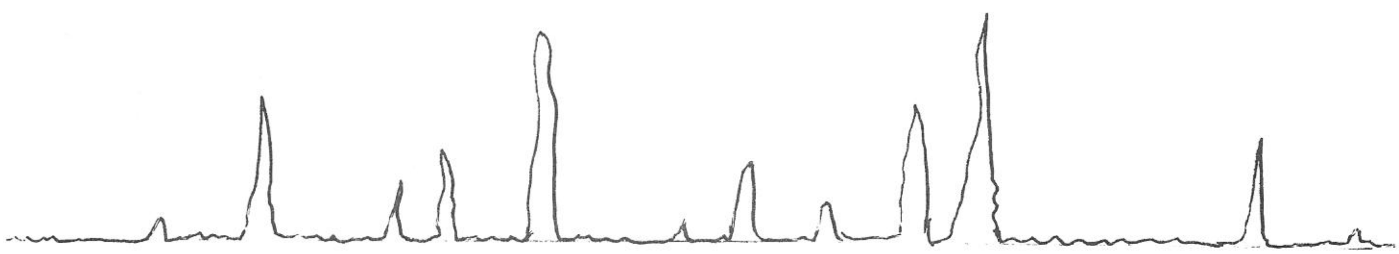


FIGURE 1 (Example of Ultrasonic Trace)

Each peak (or indication) represents an inhomogeneity encountered in the material. Therefore, the number of indications is a count of the number of inhomogeneities encountered. Furthermore, the higher a peak (or indication) is, the larger the extent of the inhomogeneity (i.e., the defect diameter) that produced the peak. It can be said that indication heights are directly proportional to defect diameters. As a measure of the TOTAL DEFECT VOLUME we can take

$$V = \text{TOTAL DEFECT VOLUME} \propto N\ H^3 \tag{11}$$

where N = Total number of indications in the trace

H = Median indication height

If we take the prediction equation (10) and put D_1 proportional to H we will obtain

$$\text{LIFE} = \frac{\text{constant}}{N^{1/b}\, H^{3/b}} \tag{12}$$

This last equation (12) is the basic equation for predicting life from ultrasonic indications.

Three quantities are needed:

FIRST: The total number of indications N

SECOND: The median indication height H

THIRD: The Weibull slope of the life distribution b

ULTRASONIC ENTROPY THEORY

Final Entropy Level = Entropy at Failure = $\log_e \dfrac{1}{P(X_F)}$

X_F = Age at Failure ; $P(X_F)$ = Probability of Surviving to Age X_F.

NOTE: Entropy at any age X is defined mathematically by means of the logarithmic expression $\log_e \frac{1}{P(X)}$, where P(X) is the probability of surviving to age X.

Initial Entropy = $\log_e \dfrac{1}{P(X_o)}$ [X_o = Initial Age]

Entropy at Time t After X_o = $\log_e \dfrac{1}{P(t + X_o)}$

If the survivorship function is of the Weibull type, then

$$P(t + X_o) = e^{-\left(\frac{t+X_o}{\theta}\right)^b}$$

[b=Weibull Slope
θ=Characteristic Life]

Therefore, $\log_e \dfrac{1}{P(t + X_o)}$ = Entropy at Time t after X_o

$$= \left(\frac{t + X_o}{\theta}\right)^b$$

For ultrasonic indications of median height H_o and total count N_o the Weibull Theory of Size Effect yields the Life Prediction Formula

$$\theta_o = \frac{\text{constant}}{\left(N_o H_o^3\right)^{1/b}} = \frac{\hat{\theta}}{\left(N_o H_o^3\right)^{1/b}} \qquad (1)$$

θ_o = Predicted Additional Life to Infinite Entropy

Let us denote $N_oH_o^3$ by Ω_o. Then from (1):

$$\left(N_oH_o^3\right)^{1/b} = \Omega_o^{1/b} = \frac{\hat{\theta}}{\theta_o}$$

$$\text{or, } N_oH_o^3 = \Omega_o = \left(\frac{\hat{\theta}}{\theta_o}\right)^b = \text{Initial Entropy} \qquad (2)$$

Thus it follows that $\hat{\theta} = X_o$ = Initial Age (at Initial Entropy Ω_o). This means that a specimen with Entropy $\Omega_o = N_oH_o^3$ is $\hat{\theta}$ units older than a specimen with Zero Entropy.

Failure can be defined to be a state where the remaining life (to Infinite Entropy) is a certain fraction γ of the initial age.

If $\hat{\theta}$ = Initial Age

Then failure is where the entropy has attained a value Ω_t such that the predicted additional life to Infinite Entropy is $\theta_t = \gamma\hat{\theta}$. $(\gamma \geq 0)$.

After running for time t : $\theta_t = \theta_o - t = \dfrac{\hat{\theta}}{\Omega_o^{1/b}} - t$

Failure is the point at which $\dfrac{\hat{\theta}}{\Omega_t^{1/b}} = \dfrac{\hat{\theta}}{\Omega_o^{1/b}} - t = \gamma\hat{\theta}$

Hence, at time of failure t : $\dfrac{1}{\Omega_t^{1/b}} = \gamma$

$$\text{or, } \frac{1}{\Omega_t^{1/b}} = \gamma^b$$

$$\text{or, } \Omega_t = \frac{1}{\gamma^b} \qquad \text{(Critical Entropy)} \qquad (3)$$

Failure has occurred after a running time history equal to

$$L_o = \theta - \gamma\hat{\theta} \qquad (4)$$

Thus , the actual historical life to failure is

$$L_o = \frac{\hat{\theta}}{(N_o H_o^3)^{1/b}} - \gamma\hat{\theta} = \hat{\theta}\left[\frac{1}{(N_o H_o^3)^{1/b}} - \gamma\right] = \hat{\theta}\left[\frac{1}{\Omega_o^{1/b}} - \gamma\right] \qquad (5)$$

Where Ω_o = Initial Entropy $= \left(\frac{\hat{\theta}}{\theta_o}\right)^b$ (b = Webull Slope)

Entropy after running time t is equal to that entropy for which the remaining historical life is

$$L_t = L_o - t = \theta_o - \gamma\hat{\theta} - t = \theta_t - \gamma\hat{\theta} \qquad (6)$$

where $[\theta_t = \theta_o - t]$

Suppose t is equal to qL_o , where q denotes the fraction of the total historical life which has expired. $(0 \leq q \leq 1)$

Then , $L_t = L_o(1 - q)$, i.e. ,

$$\frac{\hat{\theta}}{\Omega_t^{1/b}} - \gamma\hat{\theta} = L_o(1 - q) \qquad (7)$$

$(\Omega_t$ = Entropy after running time t$)$

From (7) : $\hat{\theta}/\Omega_t^{1/b} = L_o(1 - q) + \gamma\hat{\theta}$

Therefore, $1/\Omega_t^{1/b} = \frac{L_o}{\hat{\theta}}(1 - q) + \gamma$, or $\Omega_t^{1/b} = \frac{1}{\frac{L_o}{\hat{\theta}}(1 - q) + \gamma}$ (8)

From (4) we obtain $L_o/\hat{\theta} = \theta_o/\hat{\theta} - \gamma$. substituting this in (8) we get

$$\Omega_t^{1/b} = \frac{1}{(\theta_o/\hat{\theta} - \gamma)(1-q)+\gamma} = \frac{1}{(1/\Omega_o^{1/b} - \gamma)(1-q)+\gamma} = \frac{\Omega_o^{1/b}}{1-q(1-\gamma\Omega_o^{1/b})} \qquad (9)$$

Equation (9) gives us the theoretical cumulative entropy when the fraction q of the total historical life has expired.

THEORETICAL NATURE OF ENTROPY GROWTH CURVES

Equation (9) is the theoretical growth function for $\Omega_t^{1/b}$ vs. q , i.e., for $(\text{Entropy})^{1/b}$ vs. Life Expiration.

Let us denote the entropy at q by Ω_q (instead of Ω_t). This means that when the fraction q of the total historical life has expired the entropy has grown to the value Ω_q.

The growth equation is

$$\Omega_q^{1/b} = \frac{\Omega_0^{1/b}}{1-q\left(1-\gamma\Omega_0^{1/b}\right)} \qquad (10)$$

For q = 0 (i.e., when no life has expired) (10) reduces to the identity $\Omega_0^{1/b} = \Omega_0^{1/b}$

For q = 1 (i.e., when all the life has expired) (10) tells us that

$$\Omega_1^{1/b} = \frac{\Omega_0^{1/b}}{1-1\left(1-\gamma\Omega_0^{1/b}\right)} = \frac{1}{\gamma}$$

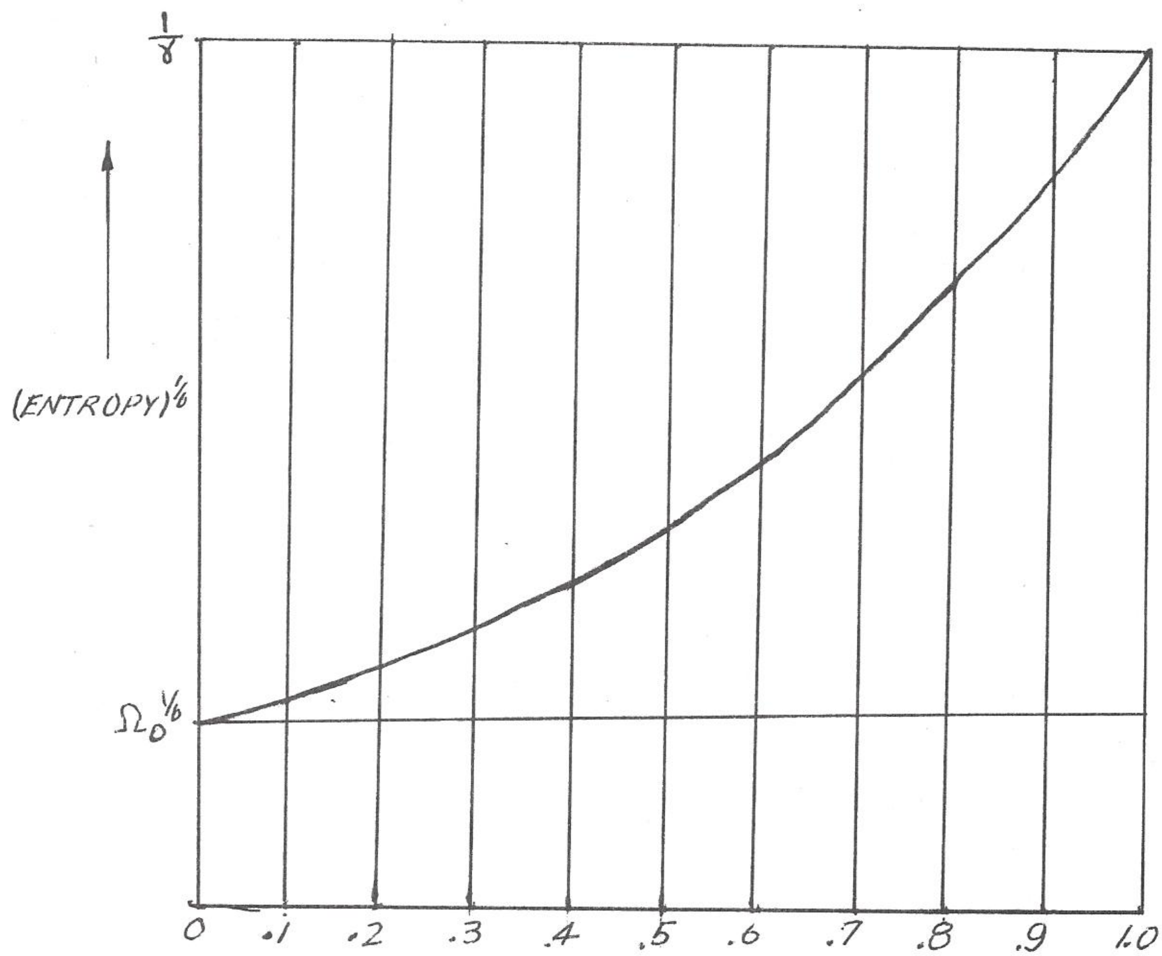
Thus, $\Omega_1 = \frac{1}{\gamma^b}$ = Critical Entropy (as it should) [see (3)]

The cumulative entropy after the <u>Half-Life</u> has expired is found by putting q = .5 in (10). This gives

$$\Omega_{.5}^{1/b} = \frac{\Omega_0^{1/b}}{1-.5\left(1-\gamma\Omega_0^{1/b}\right)}; \text{ or } \Omega_{.5} = \frac{\Omega_0}{\left[1-.5\left(1-\gamma\Omega_0^{1/b}\right)\right]^b} \qquad (11)$$

A graphical plot of (10) has the general appearance shown in Figure 3.

GENERAL APPEARANCE OF AN ENTROPY GROWTH CURVE



Expiration Fraction q

FIGURE 3

EXPRESSION FOR TOTAL AGE AT A PARTICULAR ENTROPY LEVEL

When the entropy has accumulated to the level Ω_q we know by definition that the fraction q of the total historical life L_o has expired.

Furthermore , $L_o = \theta_o - \gamma\hat{\theta}$

Therefore, at the entropy level Ω_q the historical age is

$$qL_o = q\theta_o - q\gamma\hat{\theta}$$

To get the total age we simply add the initial age $\hat{\theta}$ to the historical age. Thus, at Ω_q the <u>Total Age</u> is

$$\begin{aligned} x_q &= \hat{\theta} + qL_o = \hat{\theta} + q\theta_o - q\gamma\hat{\theta} \\ &= \hat{\theta}(1-q\gamma) + q\theta_o \\ &= \hat{\theta}\left[1 - q\gamma + q\left(\frac{\theta_o}{\hat{\theta}}\right)\right] \\ &= \hat{\theta}\left[1 - q\gamma + \frac{q}{\Omega_o^{1/b}}\right] = \hat{\theta}\left[1 - q\left(\gamma - \frac{1}{\Omega_o^{1/b}}\right)\right] \end{aligned} \qquad (12)$$

From (10) : $$\Omega_o^{1/b} = \frac{(1-q)\Omega_q^{1/b}}{1 - q\gamma\Omega_q^{1/b}} \qquad (13)$$

Substituting (13) into (12) yields the result

Total Age at Entropy $\Omega_q = x_q = \dfrac{\hat{\theta}}{(1-q)\Omega_q^{1/b}}\left[q + (1-q-\gamma q)\Omega_q^{1/b}\right]$ (14)

SLOPE OF THE ENTROPY GROWTH CURVE

THEOREM: The slope of the entropy growth curve at the time when the fraction q of the historical life has expired is given by the derivative formula

$$\frac{d\Omega_q}{dq} = \frac{b\Omega_q}{q}\left[\left(\frac{\Omega_q}{\Omega_0}\right)^{\frac{1}{b}} - 1\right] \tag{15}$$

PROOF:

From (10) $\Omega_q = \dfrac{\Omega_0}{\left[1-q\left(1-\gamma\Omega_0^{\frac{1}{b}}\right)\right]^b} = \Omega_0\left[1-q\left(1-\gamma\Omega_0^{\frac{1}{b}}\right)\right]^{-b}$

Differentiating This with respect to q yields

$$\frac{d\Omega_q}{dq} = \frac{b\Omega_0\left(1-\gamma\Omega_0^{1/b}\right)}{\left[1-q\left(1-\gamma\Omega_0^{1/b}\right)\right]^{b+1}} = \frac{b\Omega_q\left(1-\gamma\Omega_0^{1/b}\right)}{1-q\left(1-\gamma\Omega_0^{1/b}\right)}$$

Now, $1-q\left(1-\gamma\Omega_0^{1/b}\right) = \left(\dfrac{\Omega_0}{\Omega_q}\right)^{1/b}$, and $1-\gamma\Omega_0^{1/b} = \dfrac{1-\left(\frac{\Omega_0}{\Omega_q}\right)^{1/b}}{q}$

Hence , $\dfrac{d\Omega_q}{dq} = \dfrac{b\Omega_q\left[1-\left(\frac{\Omega_0}{\Omega_q}\right)^{1/b}\right]}{q\left(\frac{\Omega_0}{\Omega_q}\right)^{1/b}}$

$$= \frac{b\Omega_q}{q}\left[\left(\frac{\Omega_q}{\Omega_0}\right)^{1/b} - 1\right]$$

Q. E. D.

DERIVATIVE OF $\Omega_q^{1/b}$ WITH RESPECT TO q

Since the growth curve in Figure 3 is a plot of $\Omega_q^{1/b}$ vs. q , it is important to know the formula for $\frac{d\Omega_q^{1/b}}{dq}$, since such a formula represents the slope of the curve at any point.

By Elementary Differential Calculus:

$$\frac{d\Omega_q^{1/b}}{dq} = \frac{1}{b}\,\Omega_q^{\frac{1}{b}-1}\left(\frac{d\Omega_q}{dq}\right) = \frac{\Omega_q^{\frac{1}{b}}}{b\,\Omega_q}\left(\frac{d\Omega}{dq}\right) \qquad (16)$$

The value of $\left(\frac{d\Omega_q}{dq}\right)$ is given by (15) , i.e. ,

$$\frac{d\Omega_q}{dq} = \frac{b\Omega_q}{q}\left[\left(\frac{\Omega_q}{\Omega_0}\right)^{\frac{1}{b}} - 1\right]$$

Substituting this last expression for $\frac{d\Omega_q}{dq}$ in (16) gives

$$\frac{d\Omega_q^{\frac{1}{b}}}{dq} = \frac{\Omega_q^{\frac{1}{b}}}{q}\left(\frac{\Omega_q}{\Omega_0}\right)^{\frac{1}{b}} - 1\,\Big] \qquad (17)$$

By putting q = 0 and q = 1 , respectively, into (17) we arrive at the following formlas for initial slope and final slope in Figure 3:

$$\text{Initial Slope} = \left(\frac{d\Omega_q^{1/b}}{dq}\right)_{(q=0)} = \lim_{q\to 0}\frac{\Omega_q^{1/b}}{q}\left[\left[\frac{\Omega_q}{\Omega_0}\right]^{1/b} - 1\right] = \Omega_0^{1/b}\left(1-\gamma\Omega_0^{1/b}\right) \qquad (18)$$

$$\text{Final Slope} = \left(\frac{d\Omega_q^{1/b}}{dq}\right)_{(q=1)} = \frac{\Omega_1^{1/b}}{1}\left[\left(\frac{\Omega_1}{\Omega_0}\right)^{1/b} - 1\right] = \frac{1-\gamma\Omega_0^{1/b}}{\gamma^2\Omega_0^{1/b}} \qquad (19)$$

ANOTHER FORMULA FOR TOTAL AGE AT ENTROPY LEVEL

By definition: $\Omega_q = -\log_e P(X_q)$

X_q = Total Age at Entropy Level Ω_q

$P(X)$ = Survivorship Probability ot Age X.

For Weibull type of survivorship probability: $P(X) = e^{-\left(\frac{X}{\theta_o}\right)^b}$

Under such a Weibull Assumption: $\Omega_q = -\log_e P(X_q) = \left(\frac{X_q}{\theta_o}\right)^b$ (20)

Furthermore, the predicted additional life (to infinite entropy) is

$$\theta_q = \frac{\hat{\theta}}{\Omega_q^{1/b}} \qquad (\hat{\theta} = \text{Initial Age}) \qquad (21)$$

From (21): $\Omega_q = \left(\frac{\hat{\theta}}{\theta_q}\right)^b$ (22)

Equating (20) and (22) : $\left(\frac{X_q}{\theta_o}\right)^b = \left(\frac{\hat{\theta}}{\theta_q}\right)^b$, or $\frac{X_q}{\theta_o} = \frac{\hat{\theta}}{\theta_q}$ (23)

From (23): $X_q = \frac{\theta_o \hat{\theta}}{\theta_q} = \frac{\theta_o \hat{\theta}}{\theta_o - q(\theta_o - \gamma\hat{\theta})} = \frac{\hat{\theta}}{1 - q\left[1 - \gamma\left(\frac{\hat{\theta}}{\theta_o}\right)\right]}$

Putting $\frac{\hat{\theta}}{\theta_o} = \Omega_o^{1/b}$ in the last expression gives

$$X_q = \frac{\hat{\theta}}{1 - q(1 - \gamma\Omega_o^{1/b})} = \left[\text{Total Age at Entropy } \Omega_q\right] \qquad (24)$$

Putting q = 1 in (24) we obtain the total age at failure (i.e., when all the historical life has expired) as

$$X_1 = \frac{\hat{\theta}}{1 - 1(1 - \gamma\Omega_o^{1/b})} = \frac{\hat{\theta}}{\gamma\Omega_o^{1/b}} = \frac{\theta_o}{\gamma} \qquad (25)$$

CONFIDENCE LEVELS FOR ENTROPY AT FAILURE

The quantile level Q of failure is determined by the quantile level Q of entropy to failure. In other words, the entropy at failure (denoted by Ω_{I}) has a distribution function related to the Weibull distribution of life to failure. The Weibull function for the life of the specimen is , expressed as F(X) , for the probability of failure in running time X , is

$$F(x) = 1 - e^{-\left(\frac{x}{\theta_0}\right)^b} = Q \qquad (26)$$

Note: q and Q are not the same. q is a fraction of the quantile level of final failure.

Corresponding to this Weibull function for fatigue life we have the following cumulative distribution function for the entropy at failure:

$$Q = 1 - e^{-\frac{{}_{meas}\Omega_1}{k}} \qquad (27)$$

In this last expression ${}_{meas}\Omega_1$ represents the entropy as Measured for this reason the scale factor k is introduced. From (27) we obtain the following tabulation of Measured Entropy at failure versus the confidence, or quantile level:

Measured Entropy at Failure (${}_{meas}\Omega_1$)	Confidence Level Q	Cube Root of Measured Entropy at Failure (${}_{meas}\Omega_1^{1/3}$) ($N_1^{1/3}H_1$ in Ultrasonics)
k	.632	$k^{1/3}$
2k	.865	$1.26k^{1/3}$
2.3k	.900	$1.32k^{1/3}$
3k	.950	$1.44k^{1/3}$