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RELIABILITY OPTIMIZATION IN COMPLIANCE TESTING

Assuming N items tested annually with all of them passing.

Questions: (1) What is the optimum reliability?

(2) What is the optimum annum test sample size N?

ANSWERS

I: For a go - no go model

$$(1)\quad R_{opt.} = 1 - \sqrt{\frac{C_T}{TC_F\ (P_1 - P_0)}} \qquad \text{(Optimum Reliability)}$$

$$(2)\quad N_{opt.} = \sqrt{\frac{TC_F\ (P_1 - P_0)}{C_T}} - 2 \qquad \text{(Optimum Sample Size)}$$

C_T = Cost of testing 1 item

C_F = Cost per socially bad result

P_0 = Probability of a socially bad result for all items which comply

P_1 = Probability of a socially bad result for those items which do not comply

T = Total items produced (i.e., sold) annually

II: For A Linear Model

A linear model assumes a linear interpolation from P_0 to P_1 which is directly proportional to the fraction of the population not complying, i.e., for reliability R, the fraction not complying is (1-R), and the probability of a socially bad result for the non-complying items is

$$P_{1-R} = P_0 + (1 - R)(P_1 - P_0)$$

(Note that the probability P_1 is reached only when R = 0, i.e., only when the entire population fails to comply.)

In this case:

$$(1) \quad R_{opt.} = 1 - \sqrt[3]{\frac{C_T}{2TC_F\,(P_1 - P_0)}}$$

$$(2) \quad N_{opt.} = \sqrt[3]{\frac{2TC_F\,(P_1 - P_0)}{C_T}} - 2$$

III: For A K^{th} Power Model

In a K^{th} power model we assume the transition from P_0 to P_1 is proportional to the K^{th} power of the fraction not complying, i.e.,

$$P_{1-R} = P_0 + (1 - R)^K (P_1 - P_0)$$

(Probability of a socially bad result for each non-complying item when the population reliability is R)

In this case:

$$(1)\quad R_{opt.} = 1 - \left[\frac{C_T}{(K+1)\ T^{C}{}_{F}\ (P_1 - P_0)} \right]^{\frac{1}{K+2}}$$

$$(2)\quad N_{opt.} = \left[\frac{(K+1)\ T^{C}{}_{F}\ (P_1 - P_0)}{C_T} \right]^{\frac{1}{K+2}} - 2$$

DERIVATION OF THE OPTIMUM RELIABILITY

Our objective is to minimize the total cost where

Total cost = testing cost + social cost

Testing N items at a cost of C_T each gives a testing cost

$C_{testing} = NC_T$.

If the population reliability is R, then for T items produced there would be $T(1-R)$ which do not comply. These $T(1-R)$ each have a probability $P_{1-R} = P_0 + (1-R)^K (P_1 - P_0)$ of producing a socially bad result (such at death or disabling injury in the case of safety related compliance)

The social cost of having these $T(1-R)$ not complying is then (at cost C_F per socially bad result)

the difference

$$T(1-R)C_F P_{1-R} - T(1-R)C_F P_0 = T(1-R)C_F (P_{1-R} - P_0)$$

$$= T(1-R)^{K+1} C_F (P_1 - P_0)$$

Thus, the social cost is

$$C_{social} = T(1-R)^{K+1} C_F (P_1 - P_0)$$

Let Y = Total Cost

Then

$$Y = C_{testing} + C_{social}$$

Or

$$Y = NC_T + T(1 - R)^{K+1} \; C_F \; (P_1 - P_0)$$

For a Go - No Go Model : $K = 0$

For a Linear Model: $K = 1$

With N tested and all passing we estimate the average reliability to be

$$R = \frac{N + 1}{N + 2}$$

Hence, $N = \dfrac{2R - 1}{1 - R}$

and $$Y = C_T\left(\frac{2R - 1}{1 - R}\right) + T C_F \; (P_1 - P_0)\;(1 - R)^{K+1}$$

To find the optimum R we differentiate this total cost function with respect to R and equate the derivative to zero and solve for R.

Thus,

$$\frac{dy}{dR} = \frac{C_T}{(1 - R)^2} - T\,(K+1)\,C_F\,(P_1 - P_0)\,.(1 - R)^K = 0$$

$$\therefore \quad (1 - R)^{K+2} = \frac{C_T}{(K+1)\,T^{C}{}_F\,(P_1 - P_0)}$$

From this:

$$R_{opt.} = 1 - \left[\frac{C_T}{(K+1)\,T^{C}{}_F\,(P_1 - P_0)}\right]^{\frac{1}{K+2}}$$

Since $N = \dfrac{2R - 1}{1 - R}$ it follows

that

$$N_{opt.} = \frac{2\,R_{opt.} - 1}{1 - R_{opt.}}$$

$$= \left[\frac{(K+1)\,T^{C}{}_F\,(P_1 - P_0)}{C_T}\right]^{\frac{1}{K+2}} - 2$$

Q.E.D.

TYPICAL COST ANALYSIS FOR A SAFETY ITEM ASSUMING A LINEAR MODEL

Suppose a car driver who purchases a car from a manufacturer whose cars comply 100% with a government standard on a particular safety item has 1 chance in 30,000 annually of being killed due to impact from the particular item. Before such a standard was required there was 0% compliance, and the probability of a driver being killed in any year due to impact from the item was 1 chance in 10,000. If the annual sales for this line of cars is 1 million, and if each car test costs $4000 and each fatality costs $80,000, what is the optimum reliability level for compliance, and how large a sample should be tested annually with all cars in the sample passing?

Tabulating The Total Cost

We have $T = 10^6$, $C_T = \$4000$, $C_F = \$80{,}000$,

Let N = Test Sample Size

$$P_1 = \frac{1}{10{,}000}$$

$$P_0 = \frac{1}{30{,}000}$$

$$\text{Testing cost} = C_T N = 4000\ N$$

$$\text{Social cost} = T(1-R)^2\ C_F\ (P_1 - P_0) \quad \text{(for a linear model)}$$

For N successes in N trials: $R = \frac{N+1}{N+2}$ (avg. reliability)

$$\therefore\ 1 - R = \frac{1}{N+2} \qquad \therefore\ \text{social cost} = \frac{TC_F\,(P_1 - P_0)}{(N+2)^2}$$

WE CAN THUS FORM THE FOLLOWING COST TABLE

FOR DIFFERENT VALUES OF N (THE TEST SAMPLE SIZE)

(1)	(2)	(3)	(4)	(5)
N	4000 N C_T N	$\frac{5,333,333}{(N+2)^2}$ $\frac{T C_F (P_1 - P_0)}{(N+2)^2}$	(2)+(3)	$\frac{N+1}{N+2}$
TEST SAMPLE SIZE	TESTING COST	SOCIAL COST	TOTAL COST	RELIABILITY LEVEL
2	$8000	$333,333	$341,333	.7500
4	$16,000	$148,148	$164,148	.8333
6	$24,000	$ 83,333	$107,333	.8750
8	$32,000	$ 53,333	$ 85,333	.9000
10	$40,000	$ 37,037	$ 77,037	.9167
12 (Optimum)	$48,000	$ 27,211	$ 75,211 (Min.Cost)	.9286 (Optimum)
14	$56,000	$ 20,833	$ 76,833	.9375
16	$64,000	$ 16,461	$ 80,461	.9444
18	$72,000	$ 13,333	$ 85,333	.9500
20	$80,000	$ 11,019	$ 91,019	.9546

From this cost table we see that the optimum integral test sample size is

$$N_{opt.} = 12$$

Furthermore, the optimum reliability is

$$R_{opt.} = \frac{N_{opt.} + 1}{N_{opt.} + 2} = \frac{13}{14} = .9286$$

On page 2 we gave the formula for $R_{opt.}$ in a linear model. This was

$$R_{opt.} = 1 - \sqrt[3]{\frac{C_T}{2TC_F \; (P_1 - P_0)}}$$

$$= 1 - \sqrt[3]{\frac{4000}{2x10^6 \times 80{,}000 \left(\frac{1}{10{,}000} - \frac{1}{30{,}000}\right)}}$$

$$= 1 - \sqrt[3]{\frac{4000}{2x10^6 \times 80{,}000 \times \frac{2}{30{,}000}}}$$

$$= 1 - \sqrt[3]{\frac{12{,}000}{32{,}000{,}000}}$$

$$= 1 - \sqrt[3]{.000375} = 1 - .07211 = .92889$$

Also on page 2, the formula for $N_{opt.}$ is

$$N_{opt.} = \sqrt[3]{\frac{2T^{C}{}_{F}\,(P_1 - P_0)}{C_T}} - 2$$

$$= \sqrt[3]{\frac{2\times10^6 \times 80,000 \left(\frac{1}{10,000} - \frac{1}{30,000}\right)}{4000}} - 2$$

$$= \sqrt[3]{\frac{32,000,000}{12,000}} - 2$$

$$= \sqrt[3]{2666.667} - 2$$

$$= 13.86722 - 2$$

$$= 11.86722$$

$$= 12 \text{ (to nearest integer)}$$

The cost table thus agrees with the optimum values of N and R as found from putting $\frac{dy}{dR}$ equal to zero.

Optimal Compliance Testing When N Measurements Are Weibully Distributed With Slope b And A Maximum Of $X_N \leq X_0$ = STD.

(Assuming a K^{th} power model)

The total cost Y is given by

$$Y = NC_T + C_F T \, (P_1 - P_0) \, (1 - R)^{K+1}$$

N = Test sample size

C_T = Cost per test item

C_F = Cost per socially bad result

T = Total annual sales

P_1 = Probability of a socially bad result when the population has 0% compliance

P_0 = Probability of a socially bad result when the population has 100% compliance

R = Reliability (with regard to compliance)

K = Exponent in the assumed model, such that the probability of a socially bad result for non-compliers (when reliability = R) is

$$P_{1-R} = P_0 + (1 - R)^K \, (P_1 - P_0)$$

In case the N measurements (which are all supposed to remain below the gov't standard of X_0) have a Weibull distribution of slope b, it follows that the reliability is (on the average)

$$R = 1 - (N+1)^{-\left(\frac{X_0}{X_N}\right)^{b}} = 1 - (N+1)^{-\frac{1}{\rho^{b}}}$$

(X_N = largest of the N measurements)

$$\therefore\ 1 - R = (N+1)^{-\left(\frac{X_0}{X_N}\right)^{b}} = (N+1)^{-\frac{1}{\rho^{b}}}$$

$$\left(\rho = \frac{X_N}{X_0} \leq 1\right)$$

$$\therefore\ Y = NC_T + C_F{}^T (P_1 - P_0)(N+1)^{-\frac{K+1}{\rho^{b}}}$$

setting $\frac{dy}{dN} = 0$ we obtain

$$\frac{dy}{dN} = C_T - C_F{}^T (P_1 - P_0)\left(\frac{K+1}{\rho^{b}}\right)(N+1)^{-\frac{K+1}{\rho^{b}} - 1} = 0$$

from this we obtain

$$N_{opt.} = \left[\frac{(K+1)\, C_F T\ (P_1 - P_0)}{C_T\, \rho^{b}}\right]^{\frac{1}{1+\left(\frac{K+1}{\rho^{b}}\right)}} - 1$$

Then $R_{opt.} = 1 - (N_{opt.} + 1)^{-\frac{1}{\rho^{b}}}$

or $R_{opt.} = 1 - \left[\frac{C_T \, \rho^{b}}{(K+1) \, C_F T \, (P_1 - P_0)} \right]^{\frac{1}{K+1+\rho^{b}}}$

$$\rho = \frac{X_N}{X_0} \leq 1$$

$$\rho^{b} = \left(\frac{X_N}{X_0} \right)^{b} = \left(\frac{\theta}{X_0} \right)^{b} \ln (N + 1)$$

the relation between X_N and θ is

$$X_N = \theta \left(\ln \frac{1}{1 - \frac{N}{N+1}} \right)^{\frac{1}{b}} = \theta \ln^{\frac{1}{b}} (N + 1)$$

thus, $\quad = \dfrac{X_N}{\ln^{\frac{1}{b}} (N + 1)}$

NUMERICAL EXAMPLE

Suppose $T = 10^6$

$C_T = \$4000$ $\quad P_1 = \frac{1}{10,000}$

$C_F = \$80,000$ $\quad P_0 = \frac{1}{30,000}$

$b = 2$; $\rho = .9$ $\quad \rho^b = .81$

(Weibull Slope) (Max. observed value is 90% of STD. X_0)

Then (assuming a linear model with $K = 1$)

$$N_{opt.} = \left[\frac{2 \; C_F T \; (P_1 - P_0)}{C_T \; \rho^b}\right]^{\frac{1}{1 + \frac{2}{\rho^b}}} - 1$$

$$= \left[\frac{2 \times 80,000 \times 10^6 \left(\frac{1}{10,000} - \frac{1}{30,000}\right)}{4000 \times .81}\right]^{\frac{1}{1 + 2.46914}} - 1$$

$$= (3292.181)^{\frac{1}{3.46914}} - 1$$

$$= 10.313 - 1$$

$$= 9.313 \text{ or } 10 \text{ (to next integer)}$$

Thus, 10 of these items should be tested annually provided the largest of 10 is 90% of the Gov't spec. X_0 .

In case $N_{opt.}$ = 10 we can calculate the optimum reliability as follows:

$$R_{opt.} = 1 - (N_{opt.} + 1)^{-\frac{1}{\rho^2}}$$

$$= 1 - (11)^{-\frac{1}{.81}}$$

$$= 1 - (11)^{-1.23457}$$

$$= 1 - .0518$$

$$= .9482$$

For the fractional value of $N_{opt.}$ = 9.313 the theoretical optimum reliability $R_{opt.}$ is

$$R_{opt.} = 1 - (10.313)^{-\frac{1}{.81}}$$

$$= 1 - (10.313)^{-1.23457}$$

$$= 1 - .056093$$

$$= .943907$$