Statistical Bulletin
Reliability & Variation Research

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Vol. 1 October, 1971
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WHAT IS THE OPTIMUM LEVEL OF RELIABILITY?

The total cost of providing a product of reliability R can be separated into three basic components. These are:

(1) The Loss Due to Defectives.

This is a cost function which increases in some exponential fashion as the fraction defective increases. Thus, if Q is the fraction defective, and if Y_L is the loss (per item produced) due to defects, we can write

$$Y_L = A_L \left(\frac{Q}{1 - Q} \right)^{M_L}$$

Then, if R = Reliability = 1 - Q, this becomes

$$Y_L = A_L \left(\frac{1 - R}{R} \right)^{M_L}$$

(2) The Cost of Quality (Reliability) Design, Development and Control.

This is a cost function which increases exponentially with the reliability level R. Thus, if R is the reliability level, and Y_{QC} is the quality cost per item, we can write

$$Y_{QC} = A_{QC} \left(\frac{R}{1 - R} \right)^{M_{QC}}$$

(3) Other Production Costs per Piece not Related to Reliability.

Call this K.

Now,

Let T = Total Cost Per Item Produced.

Then,

$$T = K + Y_L + Y_{QC}$$

Or,

$$T = K + A_L \left(\frac{1 - R}{R}\right)^{M_L} + A_{QC}\left(\frac{R}{1 - R}\right)^{M_{QC}}$$

$$A_L > 0 \qquad A_{QC} > 0$$

$$M_L > 0 \qquad M_{QC} > 0$$

Let $\frac{1 - R}{R} = Z$

Then, $T = K + A_L Z^{M_L} + A_{QC} Z^{-M_{QC}}$

A_L = Loss per item produced due to defectives when the reliability level is 50%

A_{QC} = Quality Cost per item produced when the reliability level is 50%.

M_L = Exponent of the loss function due to defectives.

M_{QC} = Exponent of the quality cost function.

In order to find the optimum reliability level we differentiate T with respect to Z, and put this derivative equal to zero. Thus,

$$\frac{dT}{dZ} = A_L M_L Z^{M_L - 1} - A_{QC} M_{QC} Z^{-M_{QC} - 1} = 0$$

From this:

$$A_L M_L Z^{M_L - 1} = \frac{A_{QC} M_{QC}}{Z^{M_{QC} + 1}}$$

or, $$A_L M_L Z^{M_L + M_{QC}} = A_{QC} M_{QC}$$

$$\therefore \quad Z = \left(\frac{A_{QC} M_{QC}}{A_L M_L}\right)^{\frac{1}{M_L + M_{QC}}} = \text{Optimum Value of } Z$$

Since $Z = \frac{1 - R}{R}$, it follows that $R = \frac{1}{1 + Z}$.

Therefore, the optimum reliability level is

$$R_{opt.} = \frac{1}{1 + \left(\frac{A_{QC} M_{QC}}{A_L M_L}\right)^{\frac{1}{M_L + M_{QC}}}}$$

EXAMPLE

What is the optimum reliability (passing rate) for automobile exhaust emissions? This will depend on the parameter values A_L , M_L , A_{QC} , and M_{QC} .

We would determine

A_L = Loss per car produced due to failure to pass when the passing rate is 50%

A_{QC} = Quality Cost per car produced when the passing rate is 50%

M_L = Loss function exponent due to defective cars

M_{QC} = Quality Cost function exponent

Suppose $\left(\frac{A_L}{A_{QC}}\right) = 87.5$

Furthermore, suppose $M_L = M_{QC} = 1$.

Then, the OPTIMUM RELIABILITY LEVEL for exhaust emissions would be

$$R_{opt.} = \frac{1}{1 + \left(\frac{1}{87.5 \times 1}\right)^{\frac{1}{1+1}}} \doteq \frac{1}{1 + \sqrt{.011429}}$$

$$= \frac{1}{1.10690} = .903 \quad \text{(Ans.)}$$

Thus, in this purely academic illustration, the situation is such that the most economical passing rate is 90.3%.

DISCUSSION OF THE FUNCTION $Y_L = A_L \left(\frac{Q}{1 - Q} \right)^{M_L}$

When the failure rate is $Q = .5$, it follows that the total cost of replacing one sold item of price P (to the manufacturer) is

$$P\,(.5 + .5^2 + .5^3 + \ldots) = P = A_L$$

When the failure rate is Q, it follows that the total cost of replacing one sold item (as many times as necessary) whose price to the manufacturer is P becomes

$$P\,(Q + Q^2 + Q^3 + \ldots) = P \left(\frac{Q}{1 - Q} \right) = A_L \left(\frac{Q}{1 - Q} \right)$$

Thus, when ENTIRE ITEMS must be replaced in case of failure, it follows that $M_L = 1$.

When only the fraction γ of the price P needs to be paid by the manufacturer in case of failure, we have as total repair cost

$$\gamma P\,(Q + Q^2 + Q^3 + \ldots) = \gamma P \left(\frac{Q}{1 - Q} \right)$$

Here again, $M_L = 1$, but $A_L = \gamma P$.

DISCUSSION OF THE FUNCTION $Y_{QC} = A_{QC}\left(\frac{R}{1-R}\right)^{M_{QC}}$

The total cost of defect prevention is imbedded in the total operating cost. Many cost analysts assume

$$\text{TOTAL OPERATING COST} = \frac{\text{constant}}{Q} = \frac{A}{Q}$$

$$(Q = \text{Fraction Defective})$$

When $Q = 1$ there is no cost of defect prevention.

$$\text{Hence, COST OF DEFECT PREVENTION} = \frac{A}{Q} - A$$

$$= A\left(\frac{1}{Q} - 1\right)$$

$$= A\left(\frac{1-Q}{Q}\right)$$

$$= A\left(\frac{R}{1-R}\right) = Y_{QC}$$

Thus, $M_{QC} = 1$.

It can therefore be seen that the total cost, including all operations, as well as the correction of field defects, is

$$T = A_L\left(\frac{Q}{1-Q}\right) + \frac{A_{QC}}{Q}.$$

To find the optimum quality level we differentiate the formula for T with respect to Q, and then set this derivative equal to zero, and solve for Q.

Thus, $$\frac{dT}{dQ} = A_L \frac{(1 - Q) + Q}{(1 - Q)^2} - \frac{A_{QC}}{Q^2} = 0$$

$$\therefore \quad \frac{A_L}{(1 - Q)^2} - \frac{A_{QC}}{Q^2} = 0$$

or, $$\frac{\sqrt{A_L}}{1 - Q} = \frac{\sqrt{A_{QC}}}{Q}$$

or, $$Q\sqrt{A_L} = \sqrt{A_{QC}} - Q\sqrt{A_{QC}}$$

Solving this for the optimum level of Q yields

$$Q_{opt.} = \frac{\sqrt{A_{QC}}}{\sqrt{A_L} + \sqrt{A_{QC}}} = \frac{1}{1 + \sqrt{\frac{A_L}{A_{QC}}}}$$

Thus, the OPTIMUM RELIABILITY LEVEL is

$$R_{opt.} = 1 - Q_{opt.} = \frac{1}{1 + \sqrt{\frac{A_{QC}}{A_L}}}$$

AN APPLICATION WITH SOCIAL CONSIDERATIONS

PROBLEM: How reliable should an auto safety item be if it so happens that a fraction defective of 50% would cost extra fatalities and injuries amounting to a social cost of 400 million dollars for 8,000,000 cars sold annually, when the cost per safety item is $100 at 99.99% reliability?

SOLUTION: (Assuming $M_L = M_{QC} = 1$)

$$\text{In this case,} \quad A_L = \frac{400 \times 10^6}{8 \times 10^6} = \$50$$

$$\text{and,} \quad A_{QC} \left(\frac{.9999}{1 - .9999}\right) = 100, \text{ or } A_{QC} = \frac{100}{9999}$$

$$\text{Thus,} \quad \frac{A_{QC}}{A_L} = \frac{100}{9999 \times 50} = \frac{2}{9999} = .00020002$$

$$\text{and,} \quad R_{opt.} = \frac{1}{1 + \sqrt{\frac{A_{QC}}{A_L}}} = \frac{1}{1 + \sqrt{.00020002}} = .98605.$$

In this case the sum total of social costs and reliability costs justify a reliability level of 98.6% for this safety item.