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Reliability & Variation Research

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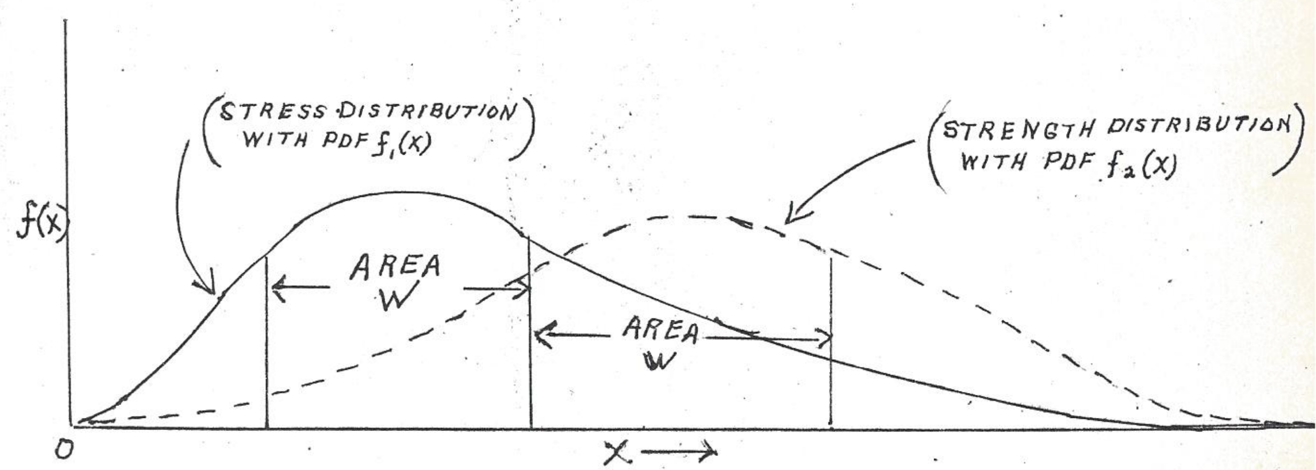
Bulletin 2

As a start for this second bulletin we shall derive the last formula given in bulletin No. 1, I.E.

$$P = \text{Probability that strength} > \text{stress} = \frac{\text{LOG}\left(\frac{1-W}{2}\right)}{\text{LOG}\left(\frac{1-W^2}{4}\right)}$$

The formula is obtained as follows:



We assume $f_1(x) = \frac{b x^{b-1}}{\theta_1^{b}} e^{-\left(\frac{x}{\theta_1}\right)^b}$ AND $f_2(x) = \frac{b x^{b-1}}{\theta_2^{b}} e^{-\left(\frac{x}{\theta_2}\right)^b}$,

Which amounts to assuming stress and strength have 2-parameter Weibull distributions of the same slope b, but different characteristic values.

Now, prob. $(x \leq \text{stress} \leq x+dx) = f_1 (x)\, dx$

prob. $\left[(\text{strength} > \text{stress}) \text{ if } (x \leq \text{stress} \leq x+dx)\right] = 1-F_2 (x)$

where $F_2 (x) = 1 - e^{-\left(\frac{x}{\theta_2}\right)^b}$

Prob. $\left[(\text{strength} > \text{stress}) \ \underline{\text{and}} \ (x \leq \text{stress} \leq x+dx)\right]$

$$= \left[1-F_2 (x)\right] f_1 (x)\, dx$$

$$= e^{-\left(\frac{x}{\theta_2}\right)^b} \cdot \frac{b x^{b-1}}{\theta_1^{b}} e^{-\left(\frac{x}{\theta_1}\right)^b} dx$$

$$= \frac{b x^{b-1}}{\theta_1^b} e^{-x^b\left(\frac{1}{\theta_1^b}+\frac{1}{\theta_2^b}\right)} dx$$

$\therefore$ Prob. $(\text{strength} > \text{stress}) = \displaystyle\int_0^\infty \frac{b x^{b-1}}{\theta_1^b} e^{-x^b\left(\frac{1}{\theta_1^b}+\frac{1}{\theta_2^b}\right)} dx$

$$= \frac{\frac{1}{\theta_1^b}}{\left(\frac{1}{\theta_1^b}+\frac{1}{\theta_2^b}\right)} \int_0^\infty b x^{b-1} \left(\frac{1}{\theta_1^b}+\frac{1}{\theta_2^b}\right) e^{-x^b\left(\frac{1}{\theta_1^b}+\frac{1}{\theta_2^b}\right)} dx$$

$$= \frac{1}{1+\left(\frac{\theta_1}{\theta_2}\right)^b} \left[-e^{-x^b\left(\frac{1}{\theta_1^b}+\frac{1}{\theta_2^b}\right)}\right]_0^\infty = \frac{1}{1+\left(\frac{\theta_1}{\theta_2}\right)^b}$$

Since central bands of width W just touch, it follows that

$$B_{\left(\frac{1+W}{2}\right)} \text{ Level of Stress} = B_{\left(\frac{1-W}{2}\right)} \text{ Level of Strength}$$

or $$\theta_1\left(\ln \frac{1}{1-\frac{1+W}{2}}\right)^{\frac{1}{b}} = \theta_2\left(\ln \frac{1}{1-\frac{1-W}{2}}\right)^{\frac{1}{b}}$$

or $$\theta_1^{\,b} \ln\left(\frac{2}{1-W}\right) = \theta_2^{\,b} \ln\left(\frac{2}{1+W}\right)$$

or $$\left(\frac{\theta_1}{\theta_2}\right)^{b} = \frac{\ln\left(\frac{2}{1+W}\right)}{\ln\left(\frac{2}{1-W}\right)} = \frac{\ln\left(\frac{1+W}{2}\right)}{\ln\left(\frac{1-W}{2}\right)}$$

PROB. (strength > stress) $$= \frac{1}{1+\left(\frac{\theta_1}{\theta_2}\right)^{b}} = \frac{1}{1+\frac{\ln\left(\frac{1+W}{2}\right)}{\ln\left(\frac{1-W}{2}\right)}}$$

$$= \frac{\ln\left(\frac{1-W}{2}\right)}{\ln\left(\frac{1-W}{2}\right)+\ln\left(\frac{1+W}{2}\right)}$$

$$= \frac{\ln\left(\frac{1-W}{2}\right)}{\ln\left(\frac{1-W^2}{4}\right)}$$

Q.E.D.

THE USE OF THE "ENTROPY" CONCEPT IN RELIABILITY

Let R (x) = the reliability of an item to target X.

Then we define

$$\text{"Entropy" at } X = \mathcal{E}(x) = \ln \frac{1}{R(x)} \qquad (1)$$

We call this an "entropy" because it involves the logarithm of a probability, namely, the negative logarithm of the survival probability R (x). Another name for this quantity $\mathcal{E}(x)$ would be the "cumulative hazard". Thus, our definition of "entropy at X" is the same as the "cumulative hazard at X".

$$\text{From (1):} \quad R(x) = e^{-\mathcal{E}(x)} \qquad (2)$$

or, in words,

$$\text{Reliability to target } x = e^{-(\text{Entropy at } x)}$$

Now, let F (x) = Probability of failure of the item before target x (i.e. F (x) = CDF of X)

$$\text{Then } F(x) = 1 - R(x) = 1 - e^{-\mathcal{E}(x)} \qquad (3)$$

From (3) it can be seen that a 2-parameter Weibull distribution $F(x) = 1 - e^{-\left(\frac{x}{\theta}\right)^{b}}$ is a special case in which "entropy at x" is given by $\mathcal{E}(x) = \left(\frac{x}{\theta}\right)^{b}$

PROPERTIES OF "ENTROPY"

PROPERTY 1: Since life has a CDF $F(x) = 1 - e^{-\mathcal{E}(x)}$ it follows that "entropy" has a CDF

$$F(\mathcal{E}) = 1 - e^{-\mathcal{E}}$$

Thus, "entropy" is always exponentially distributed with a Mean Value of Unity. This property of entropy can be used to estimate the additional life of an item which has survived to some time Xo.

Let Xo = time which an item has survived thus far.

Let $\mathcal{E}$ (Xo) = entropy at Xo

Since the average additional entropy to failure is 1, we add 1 more unit of entropy to the entropy at Xo, and solve for the X which has a total entropy of $\mathcal{E}$ (Xo) + 1.

Thus, $\mathcal{E}(X) = \mathcal{E}(Xo) + 1$

For example, suppose an item has survived 100 hours and has a Weibull CDF with b = 2 and θ = 150 hours. The entropy thus far is $\left(\frac{100}{150}\right)^2 = \frac{4}{9} = \mathcal{E}(Xo)$; (Xo = 100) add 1 more unit of entropy: $\frac{4}{9} + 1 = \frac{13}{9}$

Now find a larger X such that $\left(\frac{x}{150}\right)^2 = \frac{13}{9}$

$$\therefore \frac{x}{150} = \sqrt{\frac{13}{9}} = \frac{1}{3}\sqrt{13} = \frac{3.60553}{3} = 1.20185$$

x = 150 (1.20185) = 180.3 hrs.

Thus, an additional 80.3 hrs. will be expired before the average additional entropy of 1 will be reached.

PROPERTY II

If an item has survived to time Xo, at which time the entropy is $\mathcal{E}$ (Xo), we can predict with confidence C that the item will survive at least to a total time X such that

$$\mathcal{E}(x) = \mathcal{E}(xo) + \ln \frac{1}{c}.$$

For example, in the previous problem where $\left\{ \begin{array}{l} Xo = 100 \text{ hrs.} \\ b = 2 \\ \theta = 150 \text{ hrs.} \end{array} \right\}$

We can say with confidence 50% that the total life will at least be X where

$$\left(\frac{x}{150}\right)^2 = \left(\frac{100}{150}\right)^2 + \ln \frac{1}{.5} = \frac{4}{9} + .69315 = 1.03759$$

$$\therefore \quad \frac{x}{150} = \sqrt{1.03759} = 1.0186$$

$$x = 150\ (1.0186) = 152.8 \text{ hrs.}$$

with 50% confidence, the item will run at least another 52.8 hours

PARAMETER ESTIMATION USING ENTROPY

Problem: Given r failures at X_1, X_2, ... X_r and k suspended items at U_1, U_2 . . . . U_k from a 2-parameter Weibull Distribution of slope b. Estimate the value of θ using the fact that the average entropy to failure is 1.

Entropy at $X_1 = \mathcal{E}(x_1) = \left(\frac{x1}{\theta}\right)^b$

Entropy at $X_2 = \mathcal{E}(x_2) = \left(\frac{x2}{\theta}\right)^b$

$\vdots \qquad \vdots \qquad \vdots$

$$\text{Entropy at } X_r = \mathcal{E}(x_r) = \left(\frac{x_r}{\theta}\right)^b$$

$$\text{Entropy at } U_1 = \mathcal{E}(u_1) = \left(\frac{u_1}{\theta}\right)^b$$

$$\text{Entropy at } U_2 = \mathcal{E}(u_2) = \left(\frac{u_2}{\theta}\right)^b$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$\text{Entropy at } U_k = \mathcal{E}(u_k) = \left(\frac{u_k}{\theta}\right)^b$$

$$\text{Sum of entropies} = \left(\frac{x_1}{\theta}\right)^b + \left(\frac{x_2}{\theta}\right)^b + \dots + \left(\frac{x_r}{\theta}\right)^b + \left(\frac{u_1}{\theta_1}\right)^b + \left(\frac{u_2}{\theta}\right)^2 + \dots + \left(\frac{u_k}{\theta}\right)^b$$

Since entropies to failure are exponentially distributed with an average of unity, it follows that

$$\text{Ave. entropy to failure} = \frac{\text{sum of entropies thus far}}{\text{total failures thus far}} = 1$$

$$\text{Thus, } \frac{\left(\frac{x_1}{\theta}\right)^b + \left(\frac{x_2}{\theta}\right)^b + \dots + \left(\frac{x_r}{\theta}\right)^b + \left(\frac{u_1}{\theta}\right)^b + \left(\frac{u_2}{\theta}\right)^b + \dots + \left(\frac{u_k}{\theta}\right)^b}{r} = 1$$

$$\text{or} \quad \theta^b = \frac{x_1^b + x_2^b + \dots + x_r^b + u_1^b + u_2^b + \dots + u_k^b}{r}$$

$$\text{Thus,} \quad \theta = \left(\frac{x_1^b + x_2^b + \dots + x_r^b + u_1^b + u_2^b + \dots + u_k^b}{r}\right)^{\frac{1}{b}}$$

This estimate of the characteristic life θ is the same as the so-called maximum likelihood estimate of θ.