Statistical Bulletin
Reliability & Variation Research

DETROIT RESEARCH INSTITUTE
P.O. BOX 36504 • GROSSE POINTE, MICHIGAN 48236 • (313) 886-7976

LEONARD G. JOHNSON
EDITOR

WANG H. YEE
DIRECTOR

Volume 9 Bulletin 6 December, 1979

PREDICTING RELIABILITY FROM SPARE PARTS ORDERS

INTRODUCTION

Spare parts inventory banks are always built up in accordance with the following basic premise:

THE LAW OF EQUALITY OF INPUT AND OUTPUT:

Over some specific time interval (t_0, t_1) a set of N spare parts ordered at time t_0 will all be used up by time t_1. The set of N spare parts ordered at time t_0 can be considered as INFLOW into the inventory bank, while the use of these spare parts for replacement purposes can be considered as OUTFLOW from the same inventory bank. In this fashion, every new order for spare parts is an INFLOW, and every replacement case is an OUTFLOW. In the long run, accumulated OUTFLOW will equal accumulated INFLOW over some time lag.

This law of equality of INPUT and OUTPUT will serve as our basis for the prediction of reliability from spare parts orders. We must distinguish between various events and their dates in order to analyze this reliability prediction problem. These events are discussed in the next section.

IMPORTANT EVENTS AND TIME INTERVALS IN PARTS REPLACEMENT ANALYSIS

The various events and time intervals which need to be considered in the analysis of spare parts sales versus reliability are:

EVENT # 1: THE DELIVERY OF A NEW CAR.
We call the date of this event the DELIVERY DATE.

EVENT # 2: THE FAILURE OF THE OLD PART.
We call the date of this event the FAILURE DATE.

EVENT # 3: THE ORDERING OF THE REPLACEMENT PART.
We call the date of this event the ORDER DATE.

EVENT # 4: THE ACTUAL INSTALLATION OF THE NEW PART.
We call the date of this event the INSTALLATION DATE.

In addition to the above events and their dates, we need the following REFERENCE DATES:

REFERENCE DATE # 1: The date of the start of production of current models.

REFERENCE DATE # 2: The analysis date, on which reliability is predicted by looking at car sales and spare parts sales accumulated to that date from the start of production of the particular model being investigated.

VARIABLES DERIVABLE FROM EVENT AND REFERENCE DATES

VARIABLE # 1: THE AGE AT FAILURE.

AGE AT FAILURE = FAILURE DATE - DELIVERY DATE = X

VARIABLE # 2: MONTHS TO DELIVERY FROM START OF CURRENT PRODUCTION.

MONTHS TO DELIVERY FROM START OF CURRENT PRODUCTION =

DELIVERY DATE - REFERENCE DATE # 1 = V

VARIABLE # 3: MONTHS TO FAILURE FROM START OF CURRENT PRODUCTION.

MONTHS TO FAILURE FROM START OF CURRENT PRODUCTION =

FAILURE DATE - REFERENCE DATE # 1 = Z

VARIABLE # 4: MONTHS TO PARTS ORDER FROM START OF CURRENT PRODUCTION.

MONTHS TO PARTS ORDER FROM START OF CURRENT PRODUCTION =

ORDER DATE - REFERENCE DATE # 1 = T

VARIABLE # 5: THE NUMBER OF PARTS FAILURES IN CURRENT MODELS OF AVERAGE AGE X.

F_X = NUMBER OF PARTS FAILURES IN CURRENT MODELS OF AVE. AGE X = NO. OF PARTS ORDERED IN MONTH $T_{AVE.}$ AFTER START OF CURRENT PRODUCTION $\times$ ALLOCATION RATIO OF PARTS FAILURES FOR CURRENT MODELS

where $T_{AVE.} = Z_{AVE.}$ + (AVE. MONTHS FROM FAILURE OF OLD PART TO NEW PARTS ORDER)

VARIABLE # 6: THE NUMBER OF UNFAILED PARTS IN USE ON CARS OF AVERAGE AGE X.

S_X = NUMBER OF UNFAILED PARTS IN USE ON CARS OF AVE. AGE X

RULES EMPLOYED IN PREDICTING RELIABILITY FROM SPARE PARTS ORDERS

FIRST SET OF RULES: (FOR CURRENT MODEL YEAR ONLY)

I. A failure occurring ($Z \leq 12$) months after start of current production is, on the average, in a car ($\frac{Z}{2}$) months old.

II. A failure occurring ($Z > 12$) months after start of current production is, on the average, in a car ($Z - 6$) months old.

III. Hence, the average car age at failure, when failure occurs Z months after start of current production, is $X = \text{MAX}\ (\frac{Z}{2}, Z - 6)$.

IV. At T months after start of current production, a spare part ordered goes to repair a failure which occurred Z months after start of current production, where Z is a random number somewhere between ($T - L$) months and ($T + U$) months.

L is a random number between 0 and 12 months.

U is a random number between 0 and 24 months.

SECOND SET OF RULES: (FOR ONE YEAR OLD MODELS)

I. A failure in a one-year-old model occurring Z months after the start of production of current models is, on the average, in a one year old model of age (Z + 6) mohths.

II. Hence, the average car age at failure, when failure occurs Z months after start of current production is X = Z + 6.

III. At T months after start of current production a spare part ordered for a one-year-old model goes to repair a failure which occurred Z months after start of current production, where Z is a random number somewhere between (T - L) months and (T + U) months.

L is a random number between 0 and 12 months.

U is a random number between 0 and 24 months.

ALLOCATION FRACTIONS OVER SEVERAL MODEL YEARS

It is quite a common thing to have the same replacement part used for several model years. When such is the case, it is incorrect to assume all replacement parts are installed in the current year's model. Instead, we must properly allocate sold replacement parts to the several model years in which they are usable. This involves the following considerations:

(a) Other things being equal, older models will require more replacements due to their added service exposure.

(b) We must be knowledgeable concerning possible high failure probabilities in particular models whenever such probabilities exceed what is predicted purely by length of service. For example, new models could very well require more replacements than old models because of some special crisis or difficulty with the new models. By the same token, some older models with special difficulties could require more replacements than what is predicted only from comparative lengths of service.

ALLOCATION FRACTIONS WITHOUT SPECIAL CRISES

Let K = The number of model years using the same replacement part

Let A_i = Allocation fraction for the model year i

$i = 1$ denotes the current model year

$i = 2$ denotes one year old models

$i = 3$ denotes two year old models

. .

. .

. .

$i = K$ denotes the oldest models using the part

MATHEMATICAL ASSUMPTIONS CONCERNING WEAROUT

I : When all the model years involved are without special crises, the allocation fraction A_i for any particular model year is equal to the fraction A_{i-1} (for the later model year) increased by a constant δ (greater than zero). This constant difference δ must be such that all K allocation fractions add up to unity , i. e. ,

$$A_1 + (A_1 + \delta) + (A_1 + 2\delta) + \cdots + [A_1 + (K-1)\delta] = 1 .$$

(NOTE: $A_1 + \delta = A_2$, $A_1 + 2\delta = A_3$, ... , $A_1 + (K-1)\delta = A_K$)

(δ is known as the annual wearout increment .)

II : When there are no special crises, a replacement part used for K model years should have the fraction 1/K of all replacements allocated to the middle year of all the K model years in question, with a fraction less than 1/K allocated to model years newer than the middle year, and a fraction greater than 1/K allocated to model years older than the middle year, with successive years showing allocation fractions increasing annually by the wearout increment δ.

For example, for $K = 3$ model years, we would have allocation fractions (A_1, A_2, A_3), with

$$A_2 = A_1 + \delta \qquad (\delta > 0)$$

$$A_3 = A_1 + 2\delta$$

and $A_1 < 1/3$; $A_2 = 1/3$; $A_3 > 1/3$.

These must be such that $A_1 + A_2 + A_3 = 1$.

In case K is even , the two middle years must have allocation fractions averaging out to 1/K .

NUMBER OF VEHICLES PRODUCED AND THE AVERAGE AGES OF THOSE IN USE

T months after the start of the current model's production :

$(T/12)P_1$ current models have been produced.
$T/2$ months is the average age of current models on the road.

P_2 one-year-old models have been produced.
$(T + 6)$ months is the average age of one-year-old models on the road.

P_3 two-year-old models have been produced .
$(T + 18)$ months is the average age of two-year-old models on the road.

P_4 three-year-old models have been produced.
$(T + 30)$ months is the average age of three-year-old-models on the road.

. .
. .
. .
. .

P_K (K-1)-year-old models have been produced.
$(T + 12K-18)$ months is the average age of (K-1)-year-old models on the road.

Therefore, at the end of the current model year , the characteristic life of the part under investigation is , by maximum likelihood estimation,

$$\hat{\theta} = \left[\frac{(6P_1)^b + (18P_2)^b + (30P_3)^b + \ldots\ldots + \left[(12K-6)P_K\right]^b}{\text{Cumulative parts sales over the years 1 through } K} \right]^{1/b}$$

(b = Weibull slope.)

DETERMINATION OF $A_1(T)$ --i.e., THE CURRENT MODEL'S ALLOCATION IN MONTH #T IN ITS MODEL YR.

If a wear-out process is involved, $A_1(T)$ will be a function of T, the number of months after start of current production. If no wear-out is involved, but only random outside influences cause failure, then $A_1(T)$ will be a constant, independent of T. We can write the ratio

$$\frac{A_i(T)}{A_1(T)} = \left[\frac{T + 12i - 18}{T/2}\right]^{b-1} \text{EXP}\left[\left(\frac{T/2}{\hat{\theta}}\right)^b - \left(\frac{T + 12i - 18}{\hat{\theta}}\right)^b\right]$$

If the part has RAYLEIGH WEAR-OUT (b = 2), and if $\hat{\theta}$ is large compared to T, then

$$\frac{A_i(T)}{A_1(T)} = \frac{T + 12i - 18}{T/2} \qquad (1)$$

At the end of the current model year (when T = 12), we obtain the following ratios from (1):

i	RATIO
2	$A_2(12)/A_1(12) = 3$
3	$A_3(12)/A_1(12) = 5$
4	$A_4(12)/A_1(12) = 7$
.	.
.	.
.	.
K	$A_K(12)/A_1(12) = 2K - 1$

Hence,

$$A_1(12) + A_2(12) + A_3(12) + \cdots\cdots + A_K(12) = 1 \quad \text{becomes}$$

(Letting $A_1(12) = A_1$)

$$A_1 + 3A_1 + 5A_1 + 7A_1 + \cdots + (2K-1)A_1 = 1,$$

$$\text{or} \quad A_1 = \frac{1}{1 + 3 + 5 + 7 + \cdots + (2K-1)} = \frac{1}{K^2}$$

(For Rayleigh wear-out at 12 months)

This is in contrast to the case of zero wear-out, in which $A_1 = 1/K$.

Hence, for wear-out rates with a Weibull slope b ($1 \leq b \leq 2$), we heuristically deduce that $A_1 = 1/K^b$ (at T = 12 months). Also, for $b > 2$, we extrapolate, and take $A_1 = 1/K^b$.

Thus, for Weibull slopes 1, 1.5, 2, and 3 we can construct the following table:

ALLOCATION FRACTION	b = 1	b = 1.5	b = 2	b = 3
A_1	$1/K$	$1/K^{3/2}$	$1/K^2$	$1/K^3$
A_2	$1/K$	$(1/K^{3/2})\left(\frac{3+\sqrt{K}}{1+\sqrt{K}}\right)$	$3/K^2$	$(1/K^3)(3+2K)$
A_3	$1/K$	$(1/K^{3/2})\left(\frac{5+\sqrt{K}}{1+\sqrt{K}}\right)$	$5/K^2$	$(1/K^3)(5+4K)$
A_4	$1/K$	$(1/K^{3/2})\left(\frac{7+\sqrt{K}}{1+\sqrt{K}}\right)$	$7/K^2$	$(1/K^3)(7+6K)$
⋮	⋮	⋮	⋮	⋮
A_K	$1/K$	$(1/K^{3/2})\left(\frac{2K-1+\sqrt{K}}{1+\sqrt{K}}\right)$	$(2K-1)/K^2$	$(1/K^3)(2K^2-1)$

For the special case where K = 4 model years, this table yields the following numerical allocation fractions:

ALLOCATION FRACTION	b = 1	b = 1.5	b = 2	b = 3
A_1	.2500	.1250	.0625	.015625
A_2	.2500	.2083	.1875	.171875
A_3	.2500	.2917	.3125	.328125
A_4	.2500	.3750	.4375	.484375

Form this table we see that increasing the wear-out slope b reduces the fraction of replacements assigned to the current model, and drastically increases the number of replacements in older models.

If there are only K = 2 model years with the same part, we obtain the following table :

ALLOCATION FRACTION	b = 1	b = 1.5	b = 2	b = 3
A_1	.5000	.3536	.2500	.1250
A_2	.5000	.6464	.7500	.8750

For $K = 3$ model years with the same part, the allocation fractions are as follows:

ALLOCATION FRACTION	b = 1	b = 1.5	b = 2	b = 3
A_1	.33333	.19245	.11111	.037037
A_2	.33333	.33333	.33333	.333333
A_3	.33333	.47422	.55555	.629630

All these tables are constructed under 3 assumptions:

(1) That the analysis is at the end of the current model year ($T = 12$) .

(2) That the characteristic life $\hat{\theta}$ of the part is several times larger than $T = 12$ months (say, $\hat{\theta}$ is at least 60 months.)
(This assumption is needed because the allocation fractions assume that)
(a failure is a <u>first failure</u>.)

(3) That none of the model years with the same part have any special crises , i.e., only time in service and the wear-out slope affect comparative failure rates .

<u>REMINDER :</u> THE ABOVE NUMERICAL VALUES FOR ALLOCATION FRACTIONS ARE VALID <u>ONLY IN THE TWELFTH MONTH</u> OF THE CURRENT MODEL YEAR. The allocation fractions during other months must be developed from the ratio (1) on page 10 by using the appropriate value of T .

HOW A CRISIS AFFECTS THE CURRENT MODEL'S ALLOCATION FRACTION

The seriousness of a failure crisis (i.e., a failure epidemic) for a part in a current model is measured by the deviation of the current model's allocation fraction from what the allocation fraction normally would be in a crisis-free situation for all the K model years with the same replacement part. We can also state that in a serious crisis there is a high probability that a failure which does occur is due to the special difficulty creating the crisis, rather than being due to normal wear-out.

Let q = the probability that a failure in the current model is due to a crisis problem .

Let ${}_qA_1$ = the current model's allocation fraction when failures due to a crisis have probability q .

The normal wear-out (crisis-free) allocation fraction for the current model year is (at T = 12 months)

$$A_1(12) = 1/K^b \qquad \left(\begin{array}{l} b = \text{Weibull slope} \\ \text{(i.e., wear-out slope)} \end{array} \right)$$

We look upon this normal wear-out allocation fraction as the median rank of the first order statistic in a sample of size $\hat{N}$. $\hat{N}$ must be obtained from the equation

$$A_1 = 1/K^b = 1 - (.5)^{1/\hat{N}} .$$

Solving this for $\hat{N}$ gives

$$\hat{N} = \frac{\text{Log } .5}{\text{Log } (1 - 1/K^b)} = \frac{\text{Log } .5}{\text{Log } (1 - A_1)} .$$

Now, we define ${}_qA_1$ as the q-rank of the first order statistic in the same sample of size $\hat{N}$.

For example, if there is a 95 % probability that a failure in the current model is due to a crisis problem, then the allocation fraction for the current model becomes the 95 % rank of the first order statistic in $\hat{N}$.

As a numerical example, take the case of $K = 3$ model years with a Weibull slope 1.5. The normal allocation fractions are (at $T = 12$ mo.):

$$A_1 = .19245 , \qquad A_2 = .33333 , \qquad A_3 = .47422 .$$

Now, $A_1 = .19245$ = Median rank of 1^{st} in $\hat{N}$,

$$\text{where } \hat{N} = \frac{\text{Log } .5}{\text{Log}(1-.19245)} = \frac{\text{Log } .5}{\text{Log } .80755} = \frac{.30103}{.09283} = 3.2428 .$$

For 95 % probability of a crisis-induced failure, we then calculate

$$_{.95}A_1 = 95\ \%\ \text{rank of } 1^{st} \text{ in } 3.2428 = 1 - (1-.95)^{1/3.2428}$$
$$= 1 - (.05)^{1/3.2428} \quad 1 - .3970 = .6030 .$$

Thus, with a 95 % probability crisis in the current model, the amount allocated to the current model year jumps from a normal value of 19.245 % of all replacements over 3 model years to 60.3 % of all replacements over 3 model years. Notice that a failure is not considered to be due to a special crisis unless the odds are better than even that it is more frequent than a normal wear-out failure.

ESTIMATING THE OVERALL WEAR-OUT SLOPE AND RELIABILITY

Estimating the over-all wear-out and reliability is best described by an actual numerical example. Suppose there are K = 3 model years with the same replacement part. For example, let the cumulative number of units which have been produced in each of the 12 months of the current model year (including one-year-old and two-year-old models) be as follows:

PRODUCTION TABLE

MONTH NO.	CURRENT PROD.	CURRENT CUM.	ONE-YEAR OLD CUM. PROD.	TWO-YEAR OLD CUM. PROD.	TOTAL CUM. PROD.
1	38,000	38,000	495,600	501,100	1,033,700
2	39,500	77,500			1,073,200
3	37,700	115,200			1,110,900
4	41,400	156,600			1,152,300
5	40,200	196,800			1,192,500
6	39,000	235,800			1,231,500
7	42,000	277,800			1,273,500
8	38,500	316,300			1,312,000
9	40,100	356,400			1,352,100
10	39,800	396,200			1,391,900
11	41,100	437,300			1,433,000
12	35,600	472,900			1,468,600
CURRENT YEAR TOTAL:	472,900	C_1	C_2	C_3	$(C_1 + C_2 + C_3)$

Suppose, furthermore, that the cumulative number of replacement parts ordered in each of these months during the current model year are as follows:

PARTS SALES TABLE (CURRENT MODEL YEAR)

MONTH NO.	PARTS ORDERED	CUM. NO. OF PARTS FROM PREV. YEARS	TOTAL CUM. NUMBER OF PARTS SOLD
1	525	24,560	25,085
2	1595		26,680
3	2520		29,200
4	3050		32,250
5	2910		35,160
6	3475		38,635
7	4020		42,655
8	3955		46,610
9	4150		50,760
10	4840		55,600
11	4980		60,580
12	5225		65,805

TOTAL PARTS ORDERED (LAST 12 MO.) : 41,245

PRESENT INVENTORY POLICY ASSUMPTION: INVENTORY BANK = 25 % of TOTAL SOLD TO DATE

By lining up the production totals from the PRODUCTION TABLE and the parts sales totals from the PARTS SALES TABLE with the WEIGHTED AVERAGES OF CAR AGES , we obtain the following FAILURE RATE TABLE for all three model years collectively :

FAILURE RATE TABLE

(1) MONTH NO.	(2) CUM. CAR SALES	(3) CUM. NO. PARTS SOLD	(4) CUM. NO. OF FAILURES (INSTALLATIONS)	(5) CUM. SALES WEIGHTED AVE. CAR AGE (IN MONTHS) *	(6) PERCENT FAILED
i			75 % of (3)	$\frac{C_1(i/2)+C_2(i+6)+C_3(i+18)}{C_1 + C_2 + C_3}$	(4)÷(2)
1	1,033,700	25,085	18,814	12.567	1.820 %
2	1,073,200	26,680	20,010	13.086	1.865 %
3	1,110,900	29,200	21,900	13.624	1.971 %
4	1,152,300	32,250	24,188	14.121	2.099 %
5	1,192,500	35,160	26,370	14.630	2.211 %
6	1,231,500	38,635	28,976	15.150	2.353 %
7	1,273,500	42,655	31,991	15.640	2.512 %
8	1,312,000	46,610	34,958	16.163	2.664 %
9	1,352,100	50,760	38,070	16.671	2.816 %
10	1,391,900	55,600	41,700	17.180	2.996 %
11	1,433,000	60,580	45,435	17.678	3.171 %
12	1,468,600	65,805	49,354	18.222	3.361 %

* (C_1 , C_2 , and C_3 appear in the PRODUCTION TABLE for each month.)

NOTE: INVENTORY BANK = 16,451 (at end of current model year) .

We are now ready to predict the wear-out slope and the reliability for all three years collectively from this over-all FAILURE RATE TABLE . We do this by taking the numbers in column (5) as ABSCISSAS on WEIBULL PAPER , while making the numbers in column (6) to be ORDINATES on WEIBULL PAPER. The resulting plot is shown as the PREDICTION PLOT on the next page.

From the PREDICTION PLOT we read :

WEIBULL SLOPE = 1.74 (This is the wear-out slope b .)

B_{10} LIFE of part = 35 months

MEDIAN LIFE of part = 104 months = 8 $^2/_3$ years

CHARACTERISTIC LIFE of part = 128 months = 10 2/3 years (VALUE OF $\hat{\theta}$)

RELIABILITY OF PART FOR 12 MONTHS = 1 - .016 = .984 .

PREDICTION PLOT (OVER 3 MODEL YRS.)