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A GENERAL DISTRIBUTION FUNCTION FOR FATIGUE DATA

INTRODUCTION

In many scientific investigations a graphical representation of quantitative relations turns out to be most convenient, because it permits a quick and easy visual interpretation of results. Such is the case in the statistical analysis of fatigue data. A graph of the cumulative distribution function gives us a complete picture of the life distribution under consideration. However, the construction of a cumulative distribution function from a sample of 10 to 20 failures is not an easy task, unless the function has a very simple known shape. Therefore, it has become a common practice to change scales on rectangular coordinate paper so as to make the plot of a cumulative distribution function a straight line. This is what has been done in the present discussion of a general fatigue distribution function which has applicability to a wide variety of fatigue data. The chief use of the method discussed is that the minimum life can be estimated without first specifically guessing its value.

GENERAL FATIGUE DISTRIBUTION

A very general 3-parameter unimodal fatigue distribution is defined by

$$ {}_x p_0 = e^{-\frac{1}{K}\left[\left(\frac{x}{a}\right)^b - 1\right]} \quad * \qquad (I) \qquad (x \geq a) $$

$$ {}_x q_0 = 1 - e^{-\frac{1}{K}\left[\left(\frac{x}{a}\right)^b - 1\right]} $$

$$ \frac{1}{{}_x p_0} = e^{\frac{1}{K}\left[\left(\frac{x}{a}\right)^b - 1\right]} $$

taking logarithms, we obtain

$$ \log_e \frac{1}{{}_x p_0} = \frac{1}{K}\left[\left(\frac{x}{a}\right)^b - 1\right] $$

Hence,

$$ K = \frac{\left(\frac{x}{a}\right)^b - 1}{\log_e \frac{1}{{}_x p_0}} \qquad (\text{for all } x \geq a) $$

where ${}_x p_0$ = probability that an item of age zero will survive to age x.
${}_x q_0$ = probability that an item of age zero will be failed by age x.

In particular, if $x = \theta$, i.e., the 63.2% life, then

$$ {}_x p_0 = \frac{1}{e}, \text{ and } \log_e \frac{1}{{}_x p_0} = 1, \text{ and } K = \left(\frac{\theta}{a}\right)^b - 1. $$

* NOTE: This function is a logical extension of a two-parameter Weibull, for it represents the probability that such a Weibull item will survive to age $x > a$, if the Weibull item has already survived to age a.

It can be seen that if $x = a$, then ${}_xp_o = 1$. Hence, a is the minimum life.

In case $a = 0$, we have $K = \left(\frac{\theta}{a}\right)^b - 1 = \infty$.

We may write

$${}_xp_o = e^{-\frac{\left(\frac{x}{a}\right)^b - 1}{K}} = e^{-\frac{\left(\frac{x}{a}\right)^b - 1}{\left(\frac{\theta}{a}\right)^b - 1}}$$

or,

$${}_xp_o = e^{-\frac{x^b - a^b}{\theta^b - a^b}} \qquad (II)$$

From the last equation on page 1, it can be seen that in case $a = 0$, we have ${}_xp_o = e^{-\left(\frac{x}{\theta}\right)^b}$ which is simply a Weibull Distribution.

The probability density function corresponding to ${}_xq_o = 1 - e^{-\frac{1}{K}\left[\left(\frac{x}{a}\right)^b - 1\right]}$

$${}_x\dot{q}_o = \frac{b}{Ka^b} x^{b-1} \cdot e^{-\frac{1}{K}\left[\left(\frac{x}{a}\right)^b - 1\right]}$$

is (by differentiation)

Let us differentiate this with respect to x, and locate the mode by putting the derivative equal to zero. Thus,

$$\frac{d({}_x\dot{q}_o)}{dx} = \left\{x^{b-1} \cdot e^{-\frac{1}{K}\left[\left(\frac{x}{a}\right)^b - 1\right]} \cdot \left(\frac{1}{K}\right)\frac{b}{a^b} x^{b-1} + (b-1)x^{b-2} e^{-\frac{1}{K}\left[\left(\frac{x}{a}\right)^b - 1\right]}\right\}\frac{b}{Ka^b}$$

$$= \frac{b}{Ka^b} \cdot e^{-\frac{1}{K}\left[\left(\frac{x}{a}\right)^b - 1\right]} \cdot x^{b-2}\left[x^b\left(-\frac{1}{K}\right)\frac{b}{a^b} + (b-1)\right] = 0$$

The only relevant factor is the last one in square brackets.

Thus,

$$-\frac{b}{Ka^b} x^b + (b-1) = 0$$

This yields

$$X^b = \frac{K(b-1)a^b}{b}$$

or,

$$X = aK^{\frac{1}{b}}\left(\frac{b-1}{b}\right)^{\frac{1}{b}}$$

$$= (\theta^b - a^b)\left(\frac{b-1}{b}\right)^{\frac{1}{b}} = \text{MODE } M_o .$$

We have

$$\log_e \frac{1}{{}_xp_o} = \frac{1}{K}\left[\left(\frac{x}{a}\right)^b - 1\right]$$

$$K \log_e \frac{1}{{}_xp_o} = \left(\frac{x}{a}\right)^b - 1$$

$$\left(\frac{x}{a}\right)^b = 1 + K \log_e \frac{1}{{}_xp_o}$$

$$b \log_e x - b \log_e a = \log_e\left(1 + K \log_e \frac{1}{{}_xp_o}\right)$$

Hence, there is a linear relation between $\log_{10} x$ and $\log_{10}(1 + K'\log_{10} \frac{1}{{}_xp_o})$ * in such a distribution. For a given sample, we can vary K' until a value of K' is found which most nearly gives a linear set of points. We simply plot x as abscissa and $(1 + K' \log_{10} \frac{1}{{}_xp_o})$ as ordinate on log-log paper. Table of ordinates for sample sizes up to 20 , and for K' = 20, 10 are found **in Tables** 1 & 2. These are based on Median Ranks. As a general rule it can be stated that a value of K' which is too small will make a sample plot concave upward , and a value of K' which is too large will make a sample plot concave downward. In other words, K' must be moved in the direction of the chord in any case. If the chord is up, K' must go up, and ifthe chord is down, K' must go down, in order to rectify the cumulative plot.

(*Since it is more convenient to use the base 10 for our logarithms, we use $K' = K \log_e 10$ as a parameter.)

Let us next find the life x corresponding to a given value of ${}_xp_o$.

$$ {}_xp_o = e^{-\frac{1}{K}\left[\left(\frac{x}{a}\right)^b - 1\right]} $$

$$ \log_e \frac{1}{{}_xp_o} = \frac{1}{K}\left[\left(\frac{x}{a}\right)^b - 1\right] $$

$$ \left(\frac{x}{a}\right)^b - 1 = K \log_e \frac{1}{{}_xp_o} $$

$$ \left(\frac{x}{a}\right)^b = 1 + K \log_e \frac{1}{{}_xp_o} $$

$$ \frac{x}{a} = \left(1 + K \log_e \frac{1}{{}_xp_o}\right)^{\frac{1}{b}} $$

$$ x = a\left(1 + K \log_e \frac{1}{{}_xp_o}\right)^{\frac{1}{b}} $$

$$ = \left(a^b + a^b K \log_e \frac{1}{{}_xp_o}\right)^{\frac{1}{b}} $$

$$ = \left[a^b + a^b\left(\frac{\theta^b}{a^b} - 1\right) \log \frac{1}{{}_xp_o}\right]^{\frac{1}{b}} $$

$$ = \left[a^b + \theta^b \log_e \frac{1}{{}_xp_o} - a^b \log_e \frac{1}{{}_xp_o}\right]^{\frac{1}{b}} $$

$$ = \left[\theta^b \log_e \frac{1}{{}_xp_o} + a^b\left(1 - \log_e \frac{1}{{}_xp_o}\right)\right]^{\frac{1}{b}} $$

In terms of the notation which we have adopted, we can write

$$ N_q = \left[\theta^b \log_e \frac{1}{1-q} + a^b\left(1 - \log_e \frac{1}{1-q}\right)\right]^{\frac{1}{b}} $$

In particular, the formula for the MEDIAN is

$$ N_{.5} = \left[\theta^b \log_e 2 + a^b(1 - \log_e 2)\right]^{\frac{1}{b}} $$

$$ = (.69315\,\theta^b + .30685\,a^b)^{\frac{1}{b}} $$

The simplest method of estimating the parameters of the CDF

$$_x\mathcal{P}_0 = F(x) = 1 - e^{-\frac{x^b - a^b}{\theta^b - a^b}}$$

is by graphical trials with various values of

$$K' = \left[\left(\frac{\theta}{a}\right)^b - 1\right] \log_e 10$$ on log-log paper.

When the cumulative plot is rectified, the parameter a is the intercept on the bottom horizontal axis, the parameter b is the slope of the straight line of the rectified plot, and the parameter θ is the 63.2% point on the line, i.e., the pointe whose ordinate is $(1 + K' \log_{10} e)$.

The likelihood function based on r failures out of n is

$$L = \frac{n!}{(n-r)!}\left(\frac{b}{\theta^b - a^b}\right)^r \left(\prod_{i=1}^{r} x_i^{b-1}\right) \cdot e^{-\frac{1}{\theta^b - a^b}\left[\sum_{i=1}^{r} x^b + (n-r)x_r^b - na^b\right]}$$

assuming that a and b are known, the value of θ which maximizes this likelihood function is

$$\hat{\theta}_{r,n} = \left[\frac{\sum_{i=1}^{r} x_i^b + (n-r)(x_r^b - a^b)}{r}\right]^{\frac{1}{b}}$$

FITTING EQUATION (II) TO SOME ACTUAL CASES

Case I

In the 8th Progress Report on Fatigue as compiled by A. M. Freudenthal, O. S. Yen, and G. M. Sinclair at the University of Illinois under the sponsorship of the Office of Naval Research (1948) there are some data on non-interrupted tests on SAE 1045 steel bending specimens at three different stress levels. These same data are tabulated below with stress levels as indicated:

	Test (5) (55,000 PSI.)	Test (12) (61,000 PSI.)	Test (9) (67,500 PSI.)
Failure No.	Life(Thousands of Cycles)	Life(Thousands of Cycles)	Life(Thousands of Cycles)
1	174	78	30
2	236	82	31
3	242	85	31
4	257	86	32
5	271	95	33
6	295	98	33
7	319	101	38
8	352	102	39
9	357	105	41
10	377	105	44
11	415	123	45
12	438	126	45
13	458	126	45
14	493	127	46
15	552	137	46
16	578	147	46
17	685	148	50
18	696	158	54
19	1390	162	64
20	1676	175	75

The plots for this case are shown in Figure 1. We use the value K' = 20 , i.e., K = 8.6858 . On log-log paper we plot each life as an abscissa , with an ordinate equal to $1 + K' \log_{10} \left(\frac{1}{\text{fraction survived}} \right)$. For the j^{th} failure in n we take

(FRACTION SURVIVED) = 1 - (MEDIAN RANK OF j^{th} FAILURE IN n) .

It is ususally necessary to try different values of K' until one is found which linearizes the plot. It so happens that K' = 20 linearizes this case satisfactorily. We then obtain the following information from such a plot :

1st : The parameter a = Minimum Life = Intercept of Plot on x-axis.

2nd : The parameter θ = 63.2% Life = Abscissa at the Ordinate 1 + K , (i.e. at the ordinate 1 + .43429K')

3rd : The parameter b = Slope of Linearized Plot.

For the three stress levels levels of this case we obtain the following results :

STRESS = 55,000 psi	STRESS = 61,000 psi	STRESS = 67,000 psi
a = 152(thousand) cycles	a = 71	a = 29
θ = 480(thousand) cycles	θ = 125	θ = 43
b = 2.0	b = 4.0	b = 5.8

It should be noted that once the value of K is fixed the three paramters θ , a , and b are no longer independent, for they are connected by the relation

$$\left(\frac{\theta}{a}\right)^b - 1 = K .$$

Hence, the value of b in term of θ and a is

$$b = \frac{\log (1 + K)}{\log (\theta/a)} = \frac{\log (1 + .43429 K')}{\log (\theta/a)}$$

in this case , since K' = 20 , we have

$$b = \frac{\log 9.6858}{\log (\theta/a)} = \frac{.9861355}{\log (\theta/a)}$$

By this formula we find b = 1.97 , 4.01, and 5.76 at 55,000 psi , 61,000 psi , and 67,500 psi , respectively. For any such plot , the Median Life $N_{.5}$, is the abscissa corresponding to the ordinate $1 + K' \log_{10}(1/.5)$, i.e., the ordinate 1 + .30103 K'. In this case , since K' = 20 , the median is located at the ordinate 1 + .30103 (20) = 7.0206.

Referring to the plots we find that

at 55,000 psi, $N_{.5}$ = 402,000 cycles

at 61,000 psi, $N_{.5}$ = 114,500 cycles

at 67,500 psi, $N_{.5}$ = 40,300 cycles

The plots for this case are shown in Figure 2. The value K' = 10 is used. The estimated values of the parameters are

STRESS = 105,000 psi	STRESS = 118,000 psi
a = 69 (thousand) cycles	a = 38.5 (thousand) cycles
θ = 243 (thousand) cycles	θ = 91 (thousand) cycles
b = 1.33	b = 1.95

The median lives are 198 and 78, respectively. These are located at the ordinate $1 + 10 \log_e 2 = 4.0103$.

SPECIAL THEORETICAL RESULTS

We conclude this bulletin with some special theorems which can be proved by combining the results of this bulletin and those of the report "A General Theory of S-N Diagram", by L. G. Johnson. A brief list of interconnecting formulas is given on the following page.

THEOREMS ON THE GENERAL FATIGUE DISTRIBUTION

THEOREM 1: N_q is located at the ordinate $\left[1 + K' \log_{10}\left(\frac{1}{1-q}\right)\right]$.

In particular, θ is located at the ordinate $1 + .43429\, K'$.

Also, $N_{.5}$ is located at the ordinate $1 + .30103\, K'$.

THEOREM 2: In a given fatigue test system, in which the only variable is the scalar factor representing stress amplitude, the following quantities are invariant:

(1) K'

(2) η^b, where η is any ratio of minimum life to q-life, and b is the slope of the rectified general fatigue plot.

THEOREM 3 : The slope of the rectified general fatigue plot for a test at a given stress level is directly proportional to the quantity $\left[\log\left(\frac{S}{S_E}\right)\right]^{\gamma}$, where S is the test stress , and S_E is the endurance limit stress .

COLLECTION OF IMPORTANT FORMULAS

I :

$$b = -\frac{\log\left[1 + K' \log_{10}\left(\frac{1}{1-q}\right)\right]}{\log \eta_q} \qquad \left(\eta_q = \frac{N_o}{N_q}\right)$$

$$b = -\frac{\log(1 + .30103\,K')}{\log \eta} \qquad \left(\eta = \frac{N_o}{N_{.5}}\right)$$

II :

$$K = .43429\,K' \quad , \quad \text{or} \quad K' = 2.30259\,K$$

III :

$$\eta_2^{\,b_2} = \eta_1^{\,b_1} \quad , \quad \text{or} \quad \eta_2 = \eta_1^{\left(\frac{b_1}{b_2}\right)}$$

Hence ,

$$\log \eta_2 = \left(\frac{b_1}{b_2}\right) \log \eta_1 \quad \text{or} \quad b_2 = \frac{b_1 \log \eta_1}{\log \eta_2}$$

$$\eta_1 = \left(\frac{N_{o,s_1}}{N_{q,s_1}}\right) \quad , \quad \eta_2 = \left(\frac{N_{o,s_2}}{N_{q,s_2}}\right)$$

where b_1 = slope of plot at stress S_1

b_2 = slope of plot at stress S_2

IV :

$$\frac{b_1}{b_2} = \left[\frac{\log\left(\frac{S_1}{S_E}\right)}{\log\left(\frac{S_2}{S_E}\right)}\right]^{\gamma}$$

Case II

In the 8th Progress Report on Fatigue there also appears a pair of tests on SAE 4340 steel bending specimens at two stress levels. These are tabulated below:

	Test(3) (105,000 PSI.)	Test(1) (118,000 PSI.)
Failure No.	Life(Thousands of Cycles)	Life(Thousands of Cycles)
1	76	42
2	77	46
3	89	48
4	100	49
5	103	53
6	103	59
7	111	61
8	114	67
9	184	72
10	186	75
11	186	77
12	187	84
13	197	86
14	198	88
15	204	93
16	207	98
17	233	103
18	235	107
19	241	109
20	253	118
21	266	132
22	299	206
23	335	
24	466	
25	554	

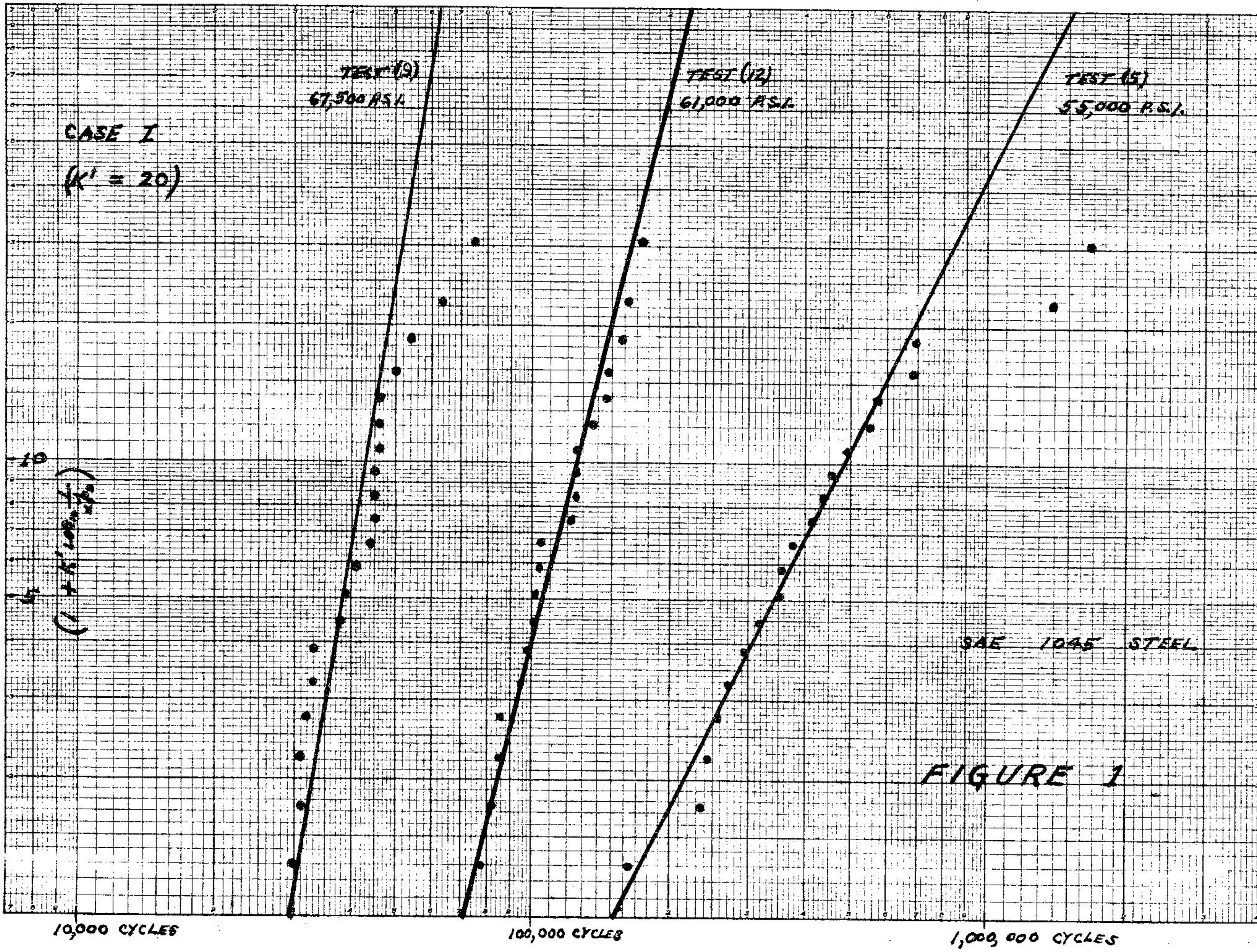
TEST (9)
67,500 P.S.I.
TEST (12)
61,000 P.S.I.
TEST (5)
55,000 P.S.I.
CASE I
(k' = 20)
10
5
SAE 1045 STEEL
10,000 CYCLES
100,000 CYCLES
1,000,000 CYCLES

FIGURE 1

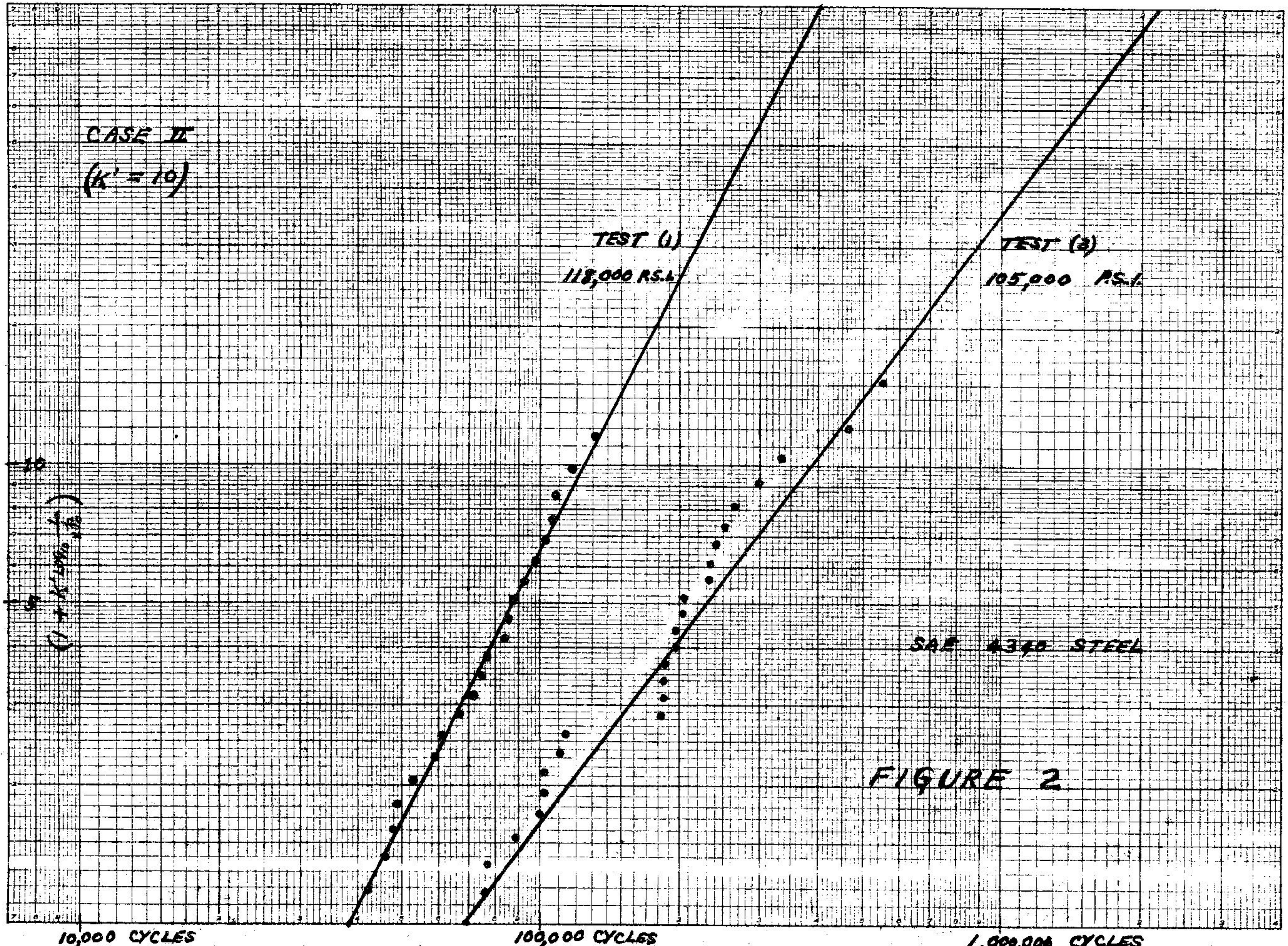
CASE II
(k' = 10)
TEST (1)
118,000 P.S.I.
TEST (2)
105,000 P.S.I.
SAE 4340 STEEL
10
5
10,000 CYCLES
100,000 CYCLES
1,000,000 CYCLES

FIGURE 2

VALUES OF $1 + 10 \log \frac{1}{s}$ (BASED ON MEDIAN RANKS)

	1	2	3	4	5	6
1	4.010	2.505	2.003	1.753	1.602	1.502
2		6.333	4.010	3.121	2.641	2.340
3			7.855	5.130	4.010	3.379
4				8.983	6.021	4.749
5					9.881	6.759
6						10.622
	7	**8**	**9**	**10**	**11**	**12**
1	1.430	1.376	1.334	1.301	1.274	1.251
2	2.132	1.981	1.865	1.774	1.700	1.639
3	2.971	2.683	2.470	2.304	2.172	2.065
4	4.010	3.521	3.172	2.909	2.703	2.538
5	5.379	4.562	4.010	3.611	3.308	3.068
6	7.392	5.931	5.051	4.450	4.010	3.673
7	11.255	7.944	6.420	5.489	4.849	4.376
8		11.809	8.433	6.860	5.889	5.214
9			12.301	8.873	7.260	6.255
10				12.739	9.271	7.625
11					13.140	9.639
12						13.510

	13	14	15	16	17	18	19	20
1	1.231	1.215	1.201	1.188	1.177	1.167	1.158	1.151
2	1.588	1.544	1.507	1.474	1.445	1.420	1.397	1.377
3	1.976	1.901	1.836	1.780	1.731	1.688	1.650	1.616
4	2.403	2.289	2.193	2.109	2.037	1.974	1.918	1.868
5	2.875	2.715	2.581	2.466	2.367	2.280	2.204	2.136
6	3.405	3.188	3.007	2.854	2.723	2.609	2.510	2.422
7	4.010	3.719	3.479	3.280	3.111	2.966	2.840	2.728
8	4.713	4.323	4.010	3.753	3.537	3.354	3.195	3.058
9	5.552	5.025	4.615	4.284	4.010	3.780	3.584	3.414
10	6.591	5.864	5.318	4.888	4.541	4.253	4.010	3.803
11	7.962	6.904	6.156	5.590	5.145	4.784	4.483	4.228
12	9.976	8.275	7.196	6.429	5.848	5.388	5.014	4.702
13	13.848	10.289	8.567	7.471	6.686	6.091	5.618	5.220
14		14.161	10.582	8.841	7.726	6.930	6.321	5.837
15			14.449	10.855	9.097	7.970	7.160	6.539
16				14.726	11.110	9.342	8.201	7.379
17					14.979	11.353	9.570	8.418
18						15.225	11.585	9.788
19							15.461	11.804
20								15.673

TABLE I

VALUES OF $(1 + 20 \log \frac{1}{s})$ (BASED ON MEDIAN RANKS)

	1	2	3	4	5	6	7	8	9	10
1	7.02	4.01	3.01	2.51	2.20	2.00	1.86	1.75	1.67	1.60
2		11.67	7.02	5.24	4.28	3.68	3.26	2.96	2.73	2.55
3			14.71	9.26	7.02	5.76	4.94	4.37	3.94	3.61
4				16.97	11.04	8.50	7.02	6.04	5.34	4.82
5					18.76	12.52	9.76	8.12	7.02	6.22
6						20.24	13.78	10.86	9.10	7.90
7							21.51	14.89	11.84	9.98
8								22.62	15.87	12.72
9									23.60	16.75
10										24.48

	11	12	13	14	15	16	17	18	19	20
1	1.55	1.50	1.46	1.43	1.40	1.38	1.35	1.33	1.32	1.30
2	2.40	2.28	2.18	2.09	2.01	1.95	1.89	1.84	1.79	1.75
3	3.34	3.13	2.95	2.80	2.67	2.56	2.46	2.38	2.30	2.23
4	4.41	4.08	3.81	3.58	3.39	3.22	3.07	2.95	2.84	2.74
5	5.62	5.14	4.75	4.43	4.16	3.93	3.73	3.56	3.41	3.27
6	7.02	6.35	5.81	5.38	5.01	4.71	4.45	4.22	4.02	3.84
7	8.70	7.75	7.02	6.44	5.96	5.56	5.22	4.93	4.68	4.46
8	10.78	9.43	8.43	7.65	7.02	6.51	6.07	5.71	5.39	5.12
9	13.52	11.51	10.10	9.05	8.23	7.57	7.02	6.56	6.17	5.83
10	17.54	14.25	12.18	10.73	9.64	8.78	8.08	7.51	7.02	6.61
11	25.28	18.28	14.92	12.81	11.31	10.18	9.29	8.57	7.97	7.46
12		26.02	18.95	15.55	13.39	11.86	10.70	9.78	9.03	8.40
13			26.70	19.58	16.13	13.94	12.37	11.18	10.24	9.44
14				27.32	20.16	16.68	14.45	12.86	11.64	10.67
15					27.90	20.71	17.19	14.94	13.32	12.08
16						28.45	21.22	17.68	15.40	13.76
17							28.96	21.71	18.14	15.84
18								29.45	22.17	18.58
19									29.92	22.61
20										30.35

TABLE II