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INCLUSIVE CONFIDENCE THEORY

INTRODUCTION

A common problem in industry and science is pictured in Figure 1 on page 2, in which a cumulative probability distribution from a sample of N order statistics is shown, together with a confidence band of width W. For W = .90, it would be a 90% confidence band around the straight line representing the median estimate of the population's cumulative distribution function. On the horizontal x-axis are two specific points, i.e., L, the lower spec., and U, the upper spec. The central (median rank) line of the diagram intersects the lower spec. L at cumulative ordinate Q_1, and intersects the upper spec. U at cumulative ordinate Q_2.

The upper boundary of the confidence band intersects L at Q_3, and intersects U at Q_6.

The lower boundary of the confidence band intersects L at Q_5, and intersects U at Q_4.

Now we ask the following questions :

I What is the confidence that the fraction of the population in the interval between L and U is at least $(Q_2 - Q_1)$?

II What is the confidence that the fraction of the population in the interval between L and U is at least $(Q_4 - Q_3)$?

III What is the confidence that the fraction of the population in the interval between L and U is at least $(Q_6 - Q_5)$?

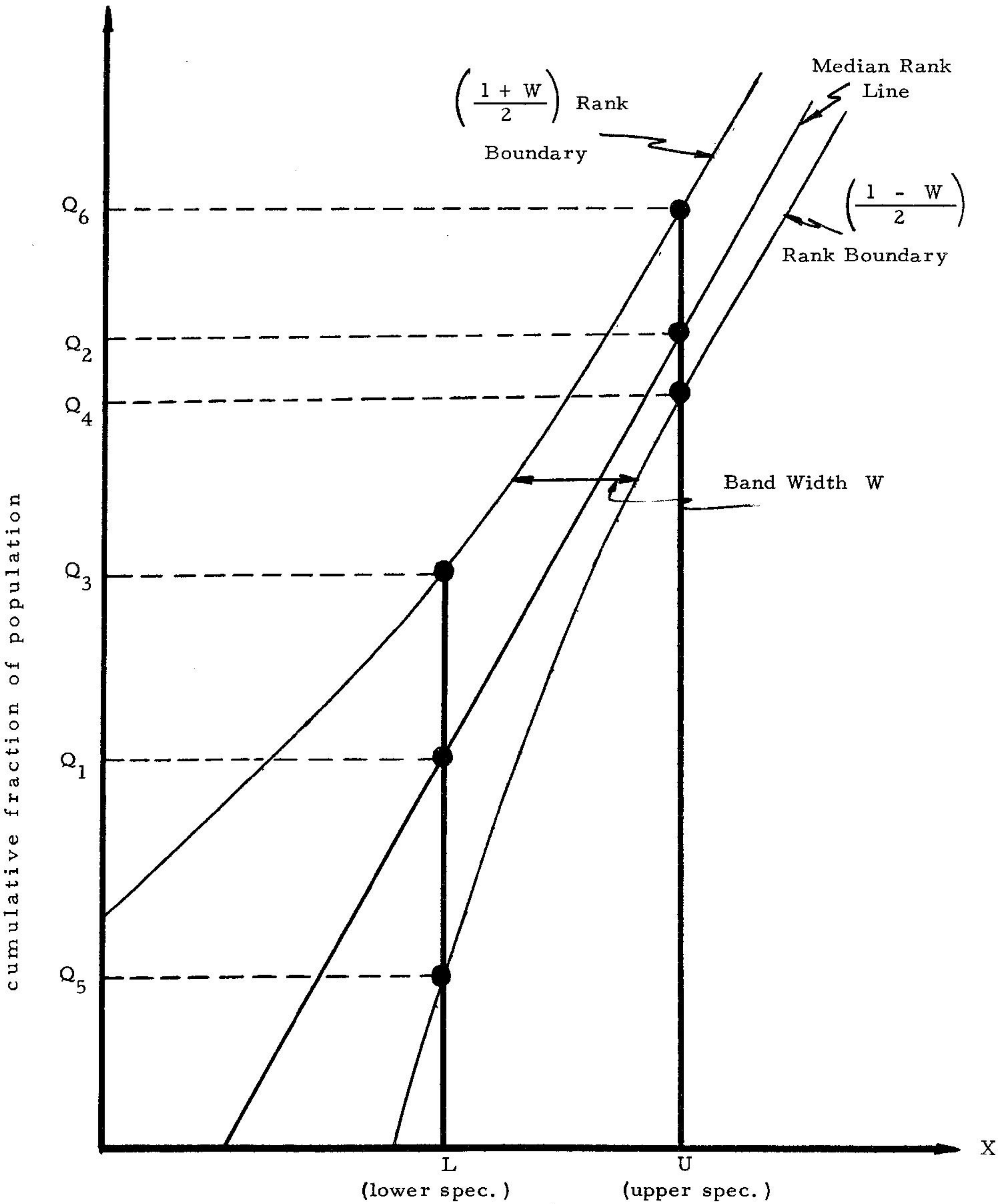
(1 + W / 2) Rank Boundary
Median Rank Line
(1 - W / 2) Rank Boundary
Band Width W
Q6
Q2
Q4
Q3
Q1
Q5
cumulative fraction of population
X
L
(lower spec.)
U
(upper spec.)

FIGURE 1

THE ANSWER TO QUESTION I

The answer to Question I is quite obvious, and presents no difficulty. Since both intersections, (Q_1 at L) and (Q_2 at U), are the Median Rank Line, it follows that the confidence that the fraction of the population included in the interval (L , U) is at least ($Q_2 - Q_1$) is simply 50% confidence.

Thus, we can construct the following table :

FRACTION OF POPULATION BETWEEN L AND U	CONFIDENCE
at least ($Q_2 - Q_1$)	50%
at most ($Q_2 - Q_1$)	50%

THE ANSWER TO QUESTION II

The Question II in the Introduction deals with the fraction ($Q_4 - Q_3$) of the population, where Q_3 is the intersection of the upper boundary with L, and Q_4 is the intersection of the lower boundary with U.

This represents a situation in which there are two neighboring confidence bands, each of width W, with a space ($Q_4 - Q_3$) between them. The confidence of a difference at least as large as ($Q_4 - Q_3$) can be calculated by employing non-overlapping band theory.
This confidence turn out to be

$$\text{CONFIDENCE} = C = \frac{\ln\left(\frac{1-W}{2}\right)}{\ln\left(\frac{1-W^2}{4}\right)}$$

Thus, we can construct the table on the next page :

FRACTION OF POPULATION BETWEEN L AND U	CONFIDENCE
at least $(Q_4 - Q_3)$	$\dfrac{\ln\left(\frac{1 - W}{2}\right)}{\ln\left(\frac{1 - W^2}{4}\right)}$
at most $(Q_4 - Q_3)$	$\dfrac{\ln\left(\frac{1 + W}{2}\right)}{\ln\left(\frac{1 - W^2}{4}\right)} = 1 - \dfrac{\ln\left(\frac{1 - W}{2}\right)}{\ln\left(\frac{1 - W^2}{4}\right)}$

THE ANSWER TO QUESTION III

Question III in the Introduction deals with the fraction $(Q_6 - Q_5)$ of the population, where Q_6 is the intersection of the upper boundary with U, and Q_5 is the intersection of the lower boundary with L.

This represents a situation in which there are two neighboring confidence bands, each of width -W, with a space $(Q_6 - Q_5)$ between them. By non-overlapping band theory, the confidence of a difference at least as large as $(Q_6 - Q_5)$ is

$$\text{CONFIDENCE} = C = \frac{\ln\left(\frac{1 + W}{2}\right)}{\ln\left(\frac{1 - W^2}{4}\right)}$$

Thus, we can construct the following table :

FRACTION OF POPULATION BETWEEN L AND U	CONFIDENCE
at least $(Q_6 - Q_5)$	$\dfrac{\ln\left(\frac{1 + W}{2}\right)}{\ln\left(\frac{1 - W^2}{4}\right)}$
at most $(Q_6 - Q_5)$	$\dfrac{\ln\left(\frac{1 - W}{2}\right)}{\ln\left(\frac{1 - W^2}{4}\right)} = 1 - \dfrac{\ln\left(\frac{1 + W}{2}\right)}{\ln\left(\frac{1 - W^2}{4}\right)}$

APPLYING THE THEORY TO AN EXAMPLE

DATA: Suppose a Weibull plot of 10 points yields Weibull parameters of $b = 1.75$ and $\theta = 500$ hrs., with 50% confidence.

PROBLEM: Find the fraction of the population which falls between 200 hrs. and 800 hrs. with (a) 50% confidence (b) 90% confidence.

SOLUTION TO (a)

The CDF is $F(x) = 1 - e^{-\left(\frac{x}{500}\right)^{1.75}}$ with 50% confidence.

Hence, $F(800) - F(200)$, which represents the fraction of the population falling between 200 hrs. and 800 hrs. is

$$F(800) - F(200) = \left[1 - e^{-\left(\frac{800}{500}\right)^{1.75}}\right] - \left[1 - e^{-\left(\frac{200}{500}\right)^{1.75}}\right]$$

$$= .89733 - .18224 = .71509$$

Thus, with 50% confidence, at least 71.509% of the population falls betwsen 200 hrs. and 800 hrs.

SOLUTION TO (b)

For 90% confidence, we need two neighboring 65% bands, constructed by using $82\frac{1}{2}\%$ ranks for the upper boundary, and using $17\frac{1}{2}\%$ ranks for the lower boundary.

This is because $$\frac{\ln\left(\frac{1 - .65}{2}\right)}{\ln\left(\frac{1 - .65^2}{4}\right)} = .90$$

Hence, according to Question II of the Introduction, the value of Q_3 is the $82\frac{1}{2}\%$ rank of the order statistic corresponding to 200 hrs., while the value of Q_4 is the $17\frac{1}{2}\%$ rank of the order statistic corresponding to 800 hrs.

200 hrs. has order statistic no. j_{200}, where

$$\frac{j_{200} - .3}{10 + .4} = 1 - e^{-\left(\frac{200}{500}\right)^{1.75}} = .18224$$

or $j_{200} = 2.19532$ (in a sample of 10)

Hence, $Q_3 = 82\frac{1}{2}\%$ rank of #2.19532 in 10 = .3089

800 hrs. has order statistic no. j_{800}, where

$$\frac{j_{800} - .3}{10 + .4} = 1 - e^{-\left(\frac{800}{500}\right)^{1.75}} = .89733$$

or $j_{800} = 9.63219$ (in a sample of 10)

Hence, $Q_4 = 17\frac{1}{2}\%$ rank of #9.63219 in 10 = .7869.

Thus, with 90% confidence, at least .7869 - .3089 = .478 = 47.8% of the population falls between 200 hours and 800 hours.