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THE PROT METHOD OF ACCELERATED FATIGUE TESTING

BASIC PROCEDURE

In the Prot Method of fatigue testing , at each new cycle of operation the stress is increased by the multiplying factor $(1 + d)$, where

d = rate of increase of stress per cycle.

Such a procedure of increasing the stress at the rate d per cycle is continued until failure occurs. The stress at failure and the cycles it takes to produce failure are then known for each item so tested. Thus , we can write

$$1 + d = \frac{\text{STRESS IN CYCLE } (j + 1)}{\text{STRESS IN CYCLE } j} \qquad \text{(for all } j\text{)}$$

MATHEMATICAL FORMULATION

Let S_o = initial stress (at start of test)

then , the stress at failure (after k cycles of operation) will be

$$S_k = S_o (1 + d)^{k-1} \qquad (1)$$

From (1) :

$$k = \frac{\ln \left[\left(\frac{S_k}{S_o} \right) (1 + d) \right]}{\ln (1 + d)}$$

Thus,

$$\text{CYCLES TO FAILURE} = \frac{\ln \left[\left(\frac{\text{FAILURE STRESS}}{\text{INITIAL STRESS}} \right) (1 + d) \right]}{\ln (1 + d)}$$

Assuming a power function "S - N" relation, we can write (for a zero endurance limit):

$$\text{CYCLES TO FAILURE} = \frac{\text{CONSTANT}}{(\text{REPRESENTATIVE STRESS})^m},$$

where

REPRESENTATIVE STRESS = ROOT MEAN m^{th} POWER STRESS (for the entire test on a specimen)

DERIVATION OF A FORMULA FOR THE REPRESENTATIVE STRESS

In a Prot test, each Stress cycle has a stress value which is one term in a Geometric Regression with a common ratio $(1 + d)$ between neighbors. If it takes k cycles to fail an item which starts at stress S_o and fails at S_k, then the sum of m^{th} powers of all the stresses making up the complete test to failure of the item is

$$\frac{(1 + d)^m S_k^m - S_o^m}{(1 + d)^m - 1}.$$

Since these are k stress levels, it follows that the Root Mean m^{th} power of all such stress is

$$\left[\frac{(1 + d)^m S_k^m - S_o^m}{k \left[(1 + d)^m - 1 \right]} \right]^{\frac{1}{m}}, \quad \text{which}$$

is the REPRESENTATIVE STRESS for the entire test.

This REPRESENTATIVE STRESS is the SINGLE STRESS which does the same amount of fatigue damage in k cycles as is done by the set of k stresses from S_o through S_k in a Prot Test.

THEORY FOR THE CONSTRUCTION OF AN "S - N" DIAGRAM FROM THE TEST DATA OBTAINED IN A PROT TEST

We can construct an "S - N" scatter diagram by taking the abscissas and ordinates generated from the following tabulated formulas :

TABLE I

ABSCISSA OF "S - N" PLOT	ORDINATE OF "S - N" PLOT
LIFE (CYCLES TO FAILURE):	REPRESENTATIVE STRESS :
k	$S_o \left[\dfrac{(1 + d)^{km} - 1}{k\left[(1 + d)^m - 1\right]} \right]^{\frac{1}{m}}$

The symbols employed are defined as follows :

S_o = initial stress

$(1 + d) = \dfrac{\text{stress in cycle } (j + 1)}{\text{stress in cycle } j}$ (for all j)

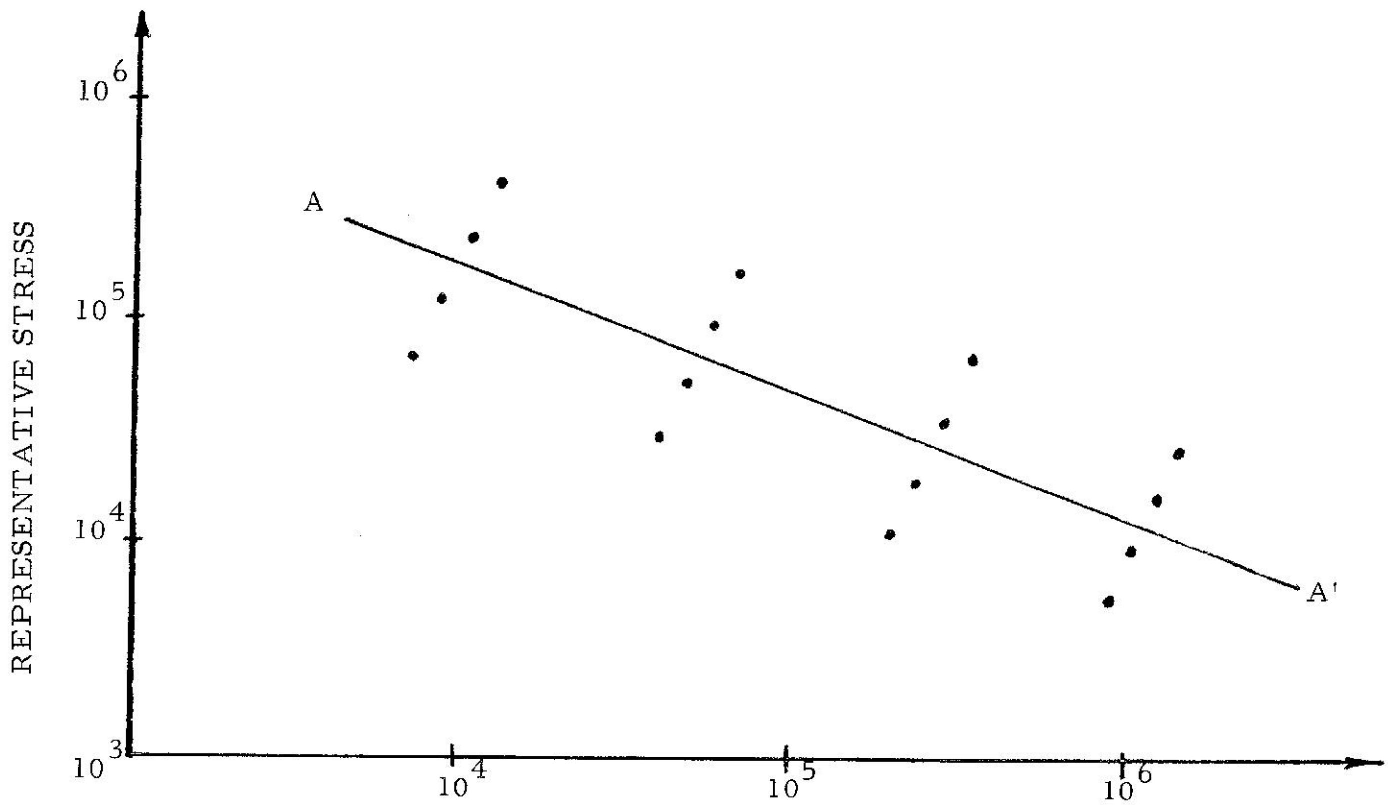
d = rate of increase of stress per cycle

$$k = \frac{\ln \left[\left(\frac{S_k}{S_o} \right) (1 + d) \right]}{\ln (1 + d)} = \text{cycles to failure}$$

S_k = stress at failure

m = "S - N" exponent

By using the abscissa and ordinate formulas of Table I for all the data on all the specimens in a Prot Test , we will get a scatter diagram , as pictured in Figure 1 below : (on log-log paper)



k = CYCLES TO FAILURE

FIGURE 1

The line AA' drawn through the array of points in Figure 1 represents the Logarithmic Least Squares Regression Line which best fits the data points. The slope of this regression line must be the same as the assumed value of m in Table I . If this is not the case , we must assume a different m , and obtain a new regression line and check again as to whether or not the slope of the new regression line agrees with the newly assumed value of m .

We continue such a trial and error process with assumed values of m , until we find an assumed value of m which leads to a regression line on log-log paper which has the same slope m as the assumed value.

Once such an agreement between an assumed m and the resulting regression m is obtained, we will have arrived at the correct "S - N" relation of the form

$$\text{CYCLES TO FAILURE} = \frac{\text{CONSTANT}}{(\text{REPRESENTATIVE STRESS})^m}$$

EXAMPLE

An accelerated fatigue test is run on a new part by using the Prot Method, with rates of stress increase per cycle ranging from .01 % to .05 %, with an initial stress of 5000 psi. The results are listed in the following table:

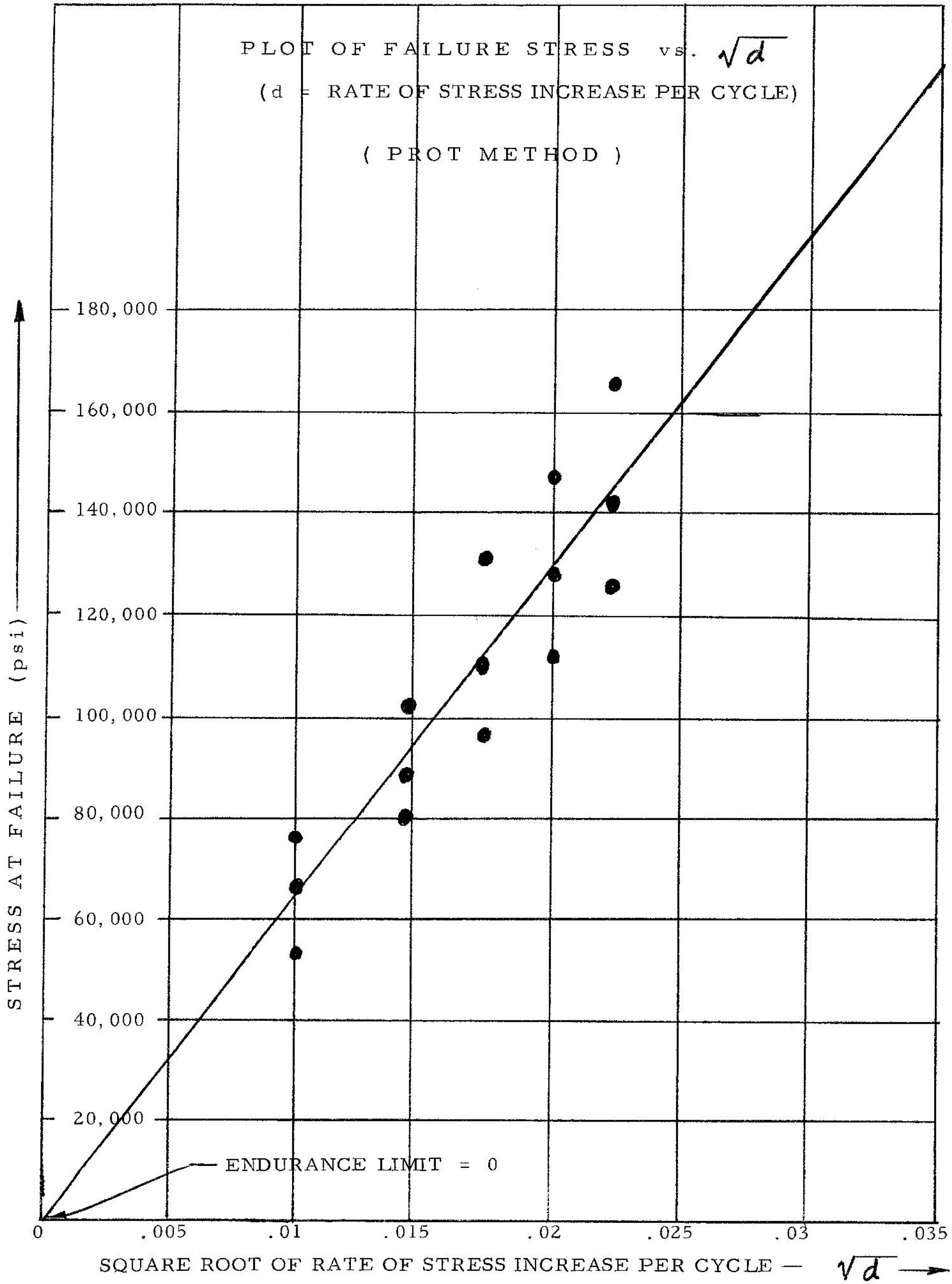
(NOTE: The Endurance Limit is Known to be ZERO.*)

TABLE II

RATE OF STRESS INCREASE /CYCLE (d)	SAMPLE SIZE	FAILURE STRESSES (psi)
.0001	3	(53, 100) (66, 200) (75, 900)
.0002	3	(80, 300) (88, 100) (102, 000)
.0003	3	(95, 200) (110, 100) (131, 400)
.0004	3	(111, 300) (128, 000) (147, 300)
.0005	3	(125, 700) (141, 000) (165, 400)

* This is verified by Figure 2.

FIGURE 2


PLOT OF FAILURE STRESS vs. $\sqrt{d}$
(d = RATE OF STRESS INCREASE PER CYCLE)
(PROT METHOD)
STRESS AT FAILURE (psi)
180,000
160,000
140,000
120,000
100,000
80,000
60,000
40,000
20,000
ENDURANCE LIMIT = 0
0
.005
.01
.015
.02
.025
.03
.035
SQUARE ROOT OF RATE OF STRESS INCREASE PER CYCLE — $\sqrt{d}$

CALCULATING THE NUMBER OF CYCLES TO FAILURE

We now take the data of Table II and calculate the number of cycles to failure for each item tested , by using the formula

$$k = \frac{\ln \left[\left(\frac{S_k}{S_o} \right) (1 + d) \right]}{\ln (1 + d)}$$

where S_o initial stress ; S_k = failure stress ;

d = rate of stress increase per cycle

Using this formula , we obtain the following table of values for k (the cycles to failure) :

TABLE III

RATE OF STRESS INCREASE/CYCLE	FAILURE STRESSES	CYCLES TO FAILURE
	53, 100	23, 630
.0001	66, 200	25, 835
	75, 900	27, 202
	80, 300	13, 884
.0002	88, 100	14, 348
	102, 000	15, 080
	95, 200	9, 824
.0003	110, 100	10, 309
	131, 400	10, 899
	111, 300	7, 760
.0004	128, 000	8, 109
	147, 300	8, 460
	125, 700	6, 452
.0005	141, 000	6, 681
	165, 400	7, 001

DETERMINING THE PARAMETERS OF THE "S - N" RELATION

Next , we assume an "S - N" relation of the form

(1) $$\text{CYCLES TO FAILURE} = \frac{\text{CONSTANT}}{(\text{REPRESENTATIVE STRESS})^m}$$

NOTE : If the endurance limit had been a number $E \neq 0$, then the above "S - N" relation would have been modified to

(2) $$\text{CYCLES TO FAILURE} = \frac{\text{CONSTANT}}{(\text{REPRESENTATIVE STRESS} - E)^m}$$

Since $E = 0$, we can simply use (1).

We do not know in advance what the true value of m is , so we guess at a value for m .

Let us take $m = 2$.

Then the REPRESENTATIVE STRESS in a test run at different stress levels is the ROOT MEAN SQUARE of all stresses used , provided $m = 2$.

In case $m = 3$, the REPRESENTATIVE STRESS must be the Root Mean Cube of all the stresses used.

In general , the REPRESENTATIVE STRESS is the ROOT MEAN m^{th} POWER of all the stresses employed in the test .

Since , in a Prot Test , the stresses in successive cycles from a Geometric Progression, it is possible to calculate the Root Mean m^{th} power of the stresses by summing a Geometric Progression of m^{th} powers , and dividing by the cycles to failure , and taking the m^{th} root of the resulting mean m^{th} power.

Thus , in general , for a Prot Test , we can write (when the endurance limit is zero) :

(3) $$\text{REPRESENTATIVE STRESS} = S_o \left\{ \frac{(1 + d)^{km} - 1}{k\left[(1 + d)^m - 1\right]} \right\}^{\frac{1}{m}}$$

Where S_o = initial stress

d = rate of stress increase per cycle

k = cycles to failure

m = "S - N" exponent

CALCULATING THE REPRESENTATIVE STRESSES

Using formula (3) on the data of Table III , we obtain the following table of representative stresses corresponding to the cycles to failure of the items in Table III (assuming m = 2) :

TABLE IV

	CYCLES TO FAILURE (k)	REPRESENTATIVE STRESS
	23, 630	24, 320
(d = .0001)	25, 835	29, 043
	27, 202	32, 472
	13, 884	34, 013
(d = .0002)	14, 348	36, 724
	15, 080	41 , 486
	9, 824	39 , 163
(d = .0003)	10, 309	44 , 233
	10, 899	51, 363
	7, 760	44, 647
(d = .0004)	8, 109	50, 230
	8, 460	56, 598
	6, 452	49, 478
(d = .0005)	6, 681	54, 528
	7 , 001	62, 518

The Logarithmic Least Squares Regression Equation from Table IV is ln (Cycles to Failure) = 26. 61269 - 1. 62093 ln (Representative Stress)

$$\text{So} \quad \text{CYCLES TO FAILURE} = \frac{36.11983}{(\text{REPRESENTATIVE STRESS})^{1.62093}} \qquad (4)$$

Thus, the value $m = 1.62093$ is obtained by logarithmic least squares by assuming $m = 2$ initially. The correct value must lie between these two values, probably nearer to 1.62. Let us try $m = 1.66$. Then Table V results.

TABLE V
(assuming $m = 1.66$)

CYCLES TO FAILURE (k)		REPRESENTATIVE STRESS
	23,630	23,033
(d = .0001)	25,835	27,314
	27,202	30,409
	13,884	31,797
(d = .0002)	14,348	34,234
	15,080	38,504
	9,824	36,424
(d = .0003)	10,309	40,962
	10,899	47,324
	7,760	41,333
(d = .0004)	8,109	46,315
	8,460	51,981
	6,452	45,645
(d = .0005)	6,681	50,142
	7,001	57,236

The Logarithmic Least Squares Regression Equation resulting from Table V

is $\ln(\text{Cycles to Failure}) = 27.12589 - 1.68093 \ln(\text{Representative Stress})$

Thus, an assumed $m = 1.660$ yielded a least squares $m = 1.681$. The correct value is probably $m = 1.677$. For all practical purposes, we can take $m = 1.68$, and use the "S - N relation

$$\text{CYCLES TO FAILURE} = \frac{e^{27.1}}{(\text{REPRESENTATIVE STRESS})^{1.68}}$$

or

$$\text{CYCLES TO FAILURE} = \frac{58.8 \times 10^{10}}{(\text{REPRESENTATIVE STRESS})^{1.68}} \quad (5)$$

The log-log plot of (5), together with the data points from Table V is given in Figure 3.

DERIVED "S - N" DIAGRAM FROM THE PROT TEST EXAMPLE

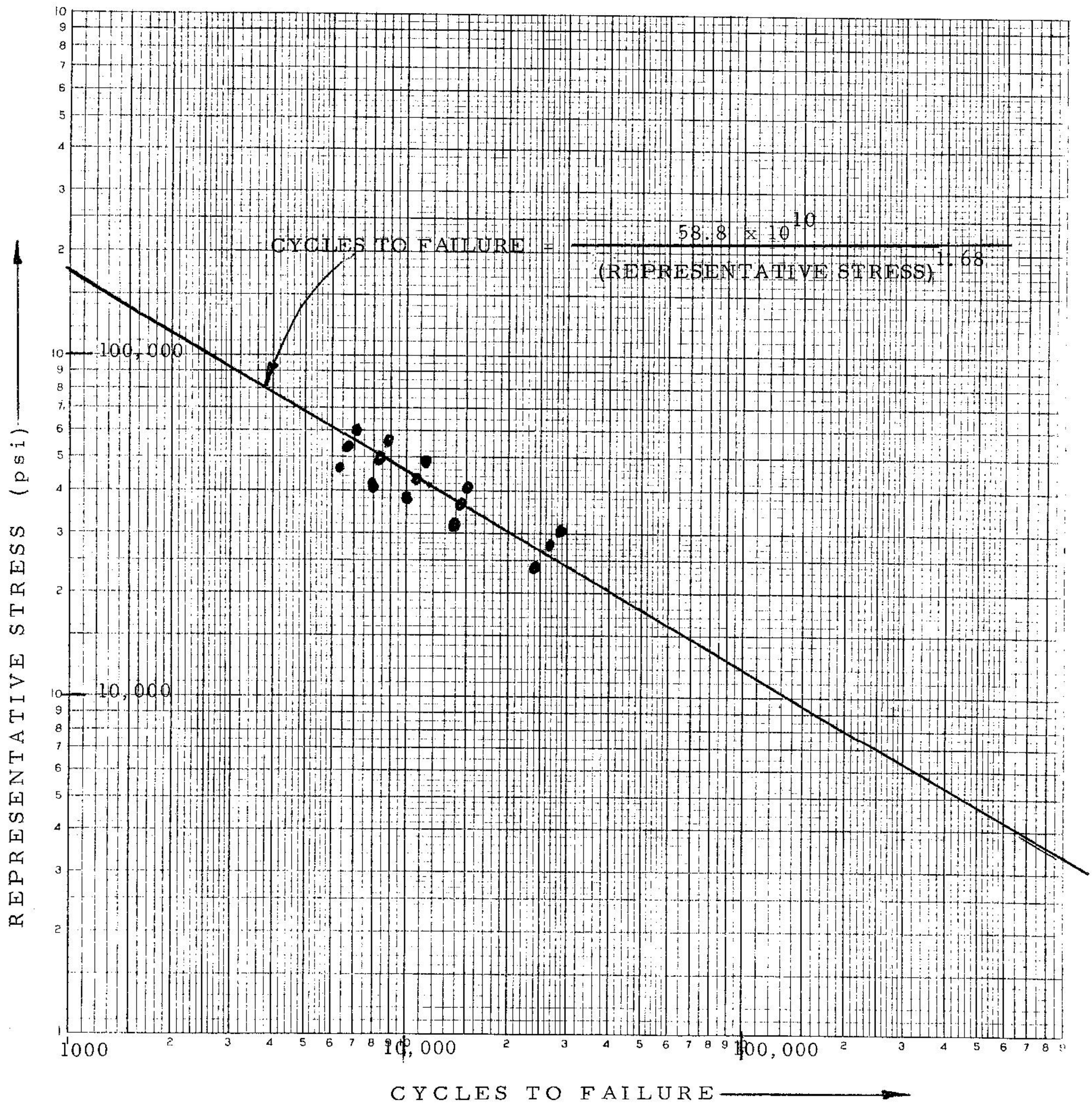


FIGURE 3