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CONFIDENCE INTERPOLATION THEORY
AND
THE GROUP THEORY OF CONFIDENCE SUPERPOSITION

CONFIDENCE INTERPOLATION THEORY

PURPOSE :

The purpose of CONFIDENCE INTERPOLATION is to make it possible to determine what CONFIDENCE can be assigned to LIFE RATIO which is between UNITY and the OBSERVED SAMPLE LIFE RATIO in a comparison test of two Weibull plots.

DEFINITIONS AND SYMBOLISM :

(A) The POPULATION LIFE RATIO for which we desire a CONFIDENCE INDEX is called the NULL RATIO .

(B) The SAMPLE LIFE RATIO between corresponding points (at equal quantile) in the two Weibull plots is called the OBSERVED LIFE RATIO.

(C) Given an OBSERVED LIFE RATIO ρ, we denote the CONFIDENCE we have in a NULL RATIO x by the symbol

$$C(x, \rho) .$$

(D) We call $C(1, \rho)$ the SIGNIFICANCE LEVEL (i. e., the CONFIDENCE of at least a UNIT POPULATION LIFE RATIO) when we observe a SAMPLE LIFE RATIO of ρ between two Weibull plots .

THE NULL RATIO CHANGE THEOREM :

The confidence for a null ratio x_1 as derived from an OBSERVED LIFE RATIO ρ (for a given Weibull slope and a given total D. F.) is equal to the confidence for a null ratio x_2 for an OBSERVED LIFE RATIO $(x_2/x_1)\rho$, for the same Weibull slope and total d. f.

Symbolically, we write this NULL RATIO CHANGE THEOREM as follows:

$$C(x_1, \rho) = C\left[x_2 , \left(\frac{x_2}{x_1}\right)\rho\right] \qquad (1)$$

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COROLLARY TO THE NULL RATIO CHANGE THEOREM:

Putting $x_2 = 1$ in (1)

$$C(x_1, \rho) = C\left(1, \frac{\rho}{x_1}\right) \qquad (2)$$

Equation (2) tells us that if we desire the confidence associated with a NULL RATIO x_1 for an OBSERVED SAMPLE LIFE RATIO ρ, we simply determine the SIGNIFICANCE INDEX for an OBSERVED LIFE RATIO (ρ/x_1) (assuming the same Weibull slope and total d.f.).

NUMERICAL EXAMPLE:

Suppose we have two neighboring Weibull plots, each of WEIBULL SLOPE 1.5 . Suppose the left hand plot has sample size 7 , and suppose the right hand WEIBULL PLOT has sample size 10 . If the OBSERVED SAMPLE MEAN LIFE RATIO is 1.75 , what is the CONFIDENCE that the ACTUAL POPULATION MEAN LIFE RATIO is at least 1.2 ?

SOLUTION :

The formula for the SIGNIFICANCE of an OBSERVED SAMPLE MEAN LIFE RATIO ρ , given a WEIBULL SLOPE b , and total d.f. T is

$$C = 1 - \frac{1}{2}\rho^{-bT^{1/4}} = C(1, \rho)$$

In this example : $\rho = 1.75$; $b = 1.5$

$$T = 6 \times 9 = 54$$

Hence , the SIGNIFICANCE LEVEL (i.e., Confidence of at least a UNIT POPULATION MEAN LIFE RATIO) is

$$C = C(1, 1.75) = 1 - \frac{1}{2}(1.75)^{-1.5(54)^{1/4}} = \underline{.94863} .$$

However, the CONFIDENCE OF AT LEAST A 1.2 POPULATION MEAN LIFE RATIO is

$$C(1.2, 1.75) = C\left(1, \frac{1.75}{1.2}\right) = C(1, 1.45833)$$

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$$C(1.2\,,1.75) = 1 - \frac{1}{2}\left(1.45833\right)^{-1.5(54)^{1/4}} = \underline{.89218}$$

The confidence to be associated with the OBSERVED SAMPLE LIFE RATIO is always 50% .

The reason why we assign 50% confidence to the observed sample life ratio is due to the fact that the sample Weibull plots are based on MEDIAN RANKS .

The SIGNIFICANCE FORMULA for an OBSERVED SAMPLE $B_{10\%}$ LIFE RATIO of magnitude ρ (assuming a Weibull slope b and TOTAL DEGREES of FREEDOM T) is

$$C = C(1\,,\,\rho) = 1 - \frac{1}{2}\rho^{-.86169\, b\, T^{1/14}} \qquad (3)$$

The test data on page 2 yield a significance index of .80889 at the $B_{10\%}$ level (use the FORMULA (3) above).

According to the test data on page 2 , we can promise with CONFIDENCE .73857 that the POPULATION $B_{10\%}$ LIFE RATIO will be at least 1.2 . (Use the NULL RATIO CHANGE THEOREM).

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THE GROUP THEORY OF CONFIDENCE SUPERPOSITION

A set of elements is called a GROUP in mathematics if, and only if, the set has the following properties:

1. Closed under some binary operation
2. A unique identity element exists
3. Each element has a unique inverse
4. The associative law holds with respect to the binary operation

Confidence numbers form a group under the binary operation of superposition.

(NOTE: A confidence number is a probability between 0 and 1.)

Thus, If C_1 is one confidence number, and C_2 is another confidence number, then their SUPERPOSITION

$$\hat{C} = \frac{C_1 C_2}{C_1 C_2 + (1 - C_1)(1 - C_2)}$$

is also a confidence number (i.e., a probability between 0 and 1).

We shall denote the superposition of C_1 and C_2 by the symbolism $C_1 \oplus C_2$.

(The binary operation of superposition is indicated by the sign $\oplus$.)

EXAMPLE: To superimpose $C_1 = .90$ and $C_2 = .80$, we calculate

$$\hat{C} = C_1 \oplus C_2 = .90 \oplus .80$$

$$= \frac{(.90)(.80)}{(.90)(.80) + (.10)(.20)} = \underline{\underline{.97297}}.$$

Thus, two confidence numbers which are both above 50% will result in a confidence which is greater than both numbers.

The IDENTITY ELEMENT in a group is that element which leaves any element unchanged when combined with the element under the binary operation.

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Thus, for the group of confidence number between 0 and 1 under superposition, the IDENTITY element is .50, since for any confidence C_1 we have $C_1 \oplus .50 = C_1$.

PROOF

$$C_1 \oplus .50 = \frac{C_1(.50)}{C_1(.50) + (1 - C_1)(1 - .50)} = C_1 .$$

Thus, C_1 is unchanged when combined with .50 under binary operation of superposition.

The INVERSE ELEMENT corresponding to a given element in a group is that element which when combined with the original element under the group's binary operation will produce the identity element as a result.

Thus the INVERSE of C_1 is $(1 - C_1)$,
because $C_1 \oplus (1 - C_1) = .50$

(The result of combining C_1 with $(1 - C_1)$ is the IDENTITY element.)

PROOF

$$C_1 \oplus (1 - C_1) = \frac{C_1(1 - C_1)}{C_1(1 - C_1) + (1 - C_1)[1 - (1 - C_1)]}$$

$$= \underline{\underline{.50}} .$$

Thus, we have proven that combining C_1 and $(1 - C_1)$ by means of the binary operation of superposition results in the IDENTITY ELEMENT, whose value is .50.

The INVERSE of 30% confidence is 70% confidence.

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What is $.30 \oplus .40$?

CALCULATION

$$.30 \oplus .40 = \frac{(.3)(.4)}{(.3)(.4) + (.7)(.6)} = \underline{\underline{.22222}}$$

Thus, superimposing two confidence numbers both less than .5 wil result in a confidence number LESS than BOTH NUMBERS.

Superimposing C_1 and C_2 can also be looked upon as a multiplication of ODDS.

$$\text{Confidence } C_1 = \text{ODDS of } \frac{C_1}{1 - C_1} = \mathcal{O}_1$$

$$\text{Confidence } C_2 = \text{ODDS of } \frac{C_2}{1 - C_2} = \mathcal{O}_2$$

Multiplying $\mathcal{O}_1$ by $\mathcal{O}_2$ gives the RESULTANT ODDS. Thus,

$$\text{RESULTANT ODDS} = \hat{\mathcal{O}} = \mathcal{O}_1\mathcal{O}_2 = \left(\frac{C_1}{1 - C_1}\right)\left(\frac{C_2}{1 - C_2}\right)$$

To convert ODDS into CONFIDENCE form the ratio

$$\frac{\text{ODDS}}{1 + \text{ODDS}}.$$

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Thus, on page 6, the resultant odds $\hat{\theta}$ can be converted into the RESULTANT CONFIDENCE $\hat{C}$ by the formula

$$\hat{C} = \frac{\hat{\theta}}{1 + \hat{\theta}}$$

or,

$$\hat{C} = \frac{\left(\frac{C_1}{1 - C_1}\right)\left(\frac{C_2}{1 - C_2}\right)}{1 + \left(\frac{C_1}{1 - C_1}\right)\left(\frac{C_2}{1 - C_2}\right)}$$

$$= \frac{C_1 C_2}{C_1 C_2 + (1 - C_1)(1 - C_2)},$$

which is the correct superposition formula.

EXAMPLE

Superimpose .9 and .8 by multiplying odds.

SOLUTION

$$\text{RESULTANT ODDS} = \frac{9}{1} \times \frac{8}{2} = \frac{72}{2} = \frac{36}{1} = \hat{\theta}$$

$$\text{RESULTANT CONFIDENCE} = \hat{C} = \frac{\hat{\theta}}{1 + \hat{\theta}} = \frac{36}{1 + 36} = \frac{36}{37}$$

$$= \underline{.97297}$$

This is the same result on page 4.

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