STATISTICAL BULLETIN

Reliability & Variation Research

LEONARD G. JOHNSON
EDITOR

DETROIT RESEARCH INSTITUTE
P.O. Box 36504 • Grosse Pointe, MI 48236 • (313) 886-8435

WANG H. YEE
DIRECTOR

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THE MATHEMATICAL CONCEPTS INVOLVED IN BASIC DURABILTY COMPLIANCE TECHNOLOGY (BDCT)

INTRODUCTION

In the modern technical age the formulation of effective mathematical systems is an absolute necessity in order to successfully complete design projects and their needed testing programs with accompanying decisions about compliance to durability goals. In this bulletin we shall emphasize the amazing usefulness of the concepts of ENTROPY and EVIDENCE in evaluating the ability of any design to comply with a desired durability goal. With the help of computer software the problems of analysis and decision making can be nicely automated so as to eliminate the mathematical errors which might occur from hand calculations made by those who are involved in the decision making process.

THE STATISTICAL CONCEPT OF ENTROPY

Whenever system durabilities are being studied over specific service periods we are concerned with the number of breakdowns (i.e., failures) in a system in those specific service periods. This statistical quantity (failures per system in a given service period) has been given the name ENTROPY for that service period. So, this term ENTROPY is just a fancy name for a FAILURE RATE over specified periods of service. For example, in the case of motor vehicles, we can talk about the number of failures per vehicle in 50,000 miles of travel. This would simply be referred to as the ENTROPY at 50,000 miles.

Obviously, ENTROPY is a quantity which must increase with service period length. Each individual system type which is being analyzed has its own cumulative distribution function which tells us the fraction of such systems failing at least once in any specific service period. The famous WEIBULL DISTRIBUTION which is used extensively nowadays in life testing was originated by Professor Weibull who took the power function $(X / \theta)^b$ as the growth formula for ENTROPY with respect to service period X, and then from actual failure data samples he determined the best fitting values of b and θ for the item tested.

The WEIBULL FORMULA for the fraction failed at least once in service period X is

$$F(X) = 1 - EXP\left[-(X / \theta)^b\right] \quad (1)$$

We can generalize this formula for the fraction failed at least once in service period X by writing it in terms of ENTROPY as

$$F(X) = 1 - EXP(- \text{ENTROPY at } X) \quad (2)$$

Let us use the notation $\mathcal{E}(X)$ = ENTROPY at X . Then, the most general expression for a cumulative distribution function of failures in any service period X would be

$$F(X) = 1 - \text{EXP}\left[-\mathcal{E}(X)\right] \qquad (3)$$

From (3): $$1 - F(X) = \text{EXP}\left[-\mathcal{E}(X)\right] \qquad (4)$$

Since F(X) is the fraction failed in service period X, it follows that 1 -F(X) is the fraction SURVIVED to service period X. We call this the RELIABILITY to service period X, and use the notaion

R(X) = Reliability to Service Period X .

Thus, from (4) : $$R(X) = \text{EXP}\left[-\mathcal{E}(X)\right] \qquad (5)$$

Solving (5) for $\mathcal{E}(X)$ we obtain

$$\mathcal{E}(X) = -\ln\left[R(X)\right] \qquad (6)$$

Thus, ENTROPY at any service period X is the absolute value of the natural logarithm of RELIABILITY to X.

EVIDENCE OF COMPLIANCE

For any system sold to customers there is what is known as the MAXIMUM TOLERABLE FAILURE RATE in any specified WARRANTY PERIOD. For example, a manufacturer might produce 1000 machines and give a warranty promise that these machines will be failure-free for at least 2000 hours of use. This means that the manufacturer must have ODDS high enough in favor of survival to 2000 hours in order to make several times as much money from surviving machines as is lost from those machines which would fail to meet the promised 2000 hours of successful operation. First and foremost, in order to design a satisfactory product, the manufacturer must know the DOLLAR GAIN per GOOD ITEM, and the DOLLAR LOSS per BAD ITEM. Then, the REQUIRED ODDS in favor of compliance to the warranty period is calculated from the formula

$$\text{ODDS} = (K * L) / G \qquad (7)$$

where G = Dollar Gain per Good Item (Complying)

L = Dollar Loss per Bad Item (Not Complying)

K = Desired Profitability Ratio , which tells how many times greater total gains will be than the total losses

We define EVIDENCE of compliance as the NATURAL LOGARITHM of ODDS in favor of complying to the pronmised warranty period. So, from (7) we can write the formula for the REQUIRED EVIDENCE of compliance in order to realize a desired profitabilty ratio K as follows:

$$\text{EVIDENCE} = \ln(\text{ODDS}) = \ln\left[(K * L) / G\right] \qquad (8)$$

Another way of looking at evidence is to ask the question, as is done in quality control :

"How many sigmas away from the nominal value is the value of the measured variable with which we are concerned?"

The evidence of being different from the nominal value is directly proportional to the number of standard deviations a measurement is away from the nominal value. This is called the Z-SCORE for the measurement. In quality control we can state that the higher the absolute value of the Z-SCORE of a measured value is the more EVIDENCE we have that the process involved is out of control. i.e., that it is away from the nominal value. In other words, EVIDENCE is proportional to the Z-SCORE. As a matter of fact, the mathematical formula for Evidence of being different from the nominal value can be written as follows:

$$\text{EVIDENCE} = (\pi / \sqrt{3}) * Z \qquad (9)$$

where Z = Z-SCORE = No. of SIGMAS the measurement is from nominal. SIGMA is defined to be the STANDARD DEVIATION in the BELL curve (i.e.,in the NORMAL DISTRIBUTION curve for the measured variable). It is customary to denote the Standard Deviation by the Greek Letter σ. Consequently, the Z-SCORE is given by the formula

$$Z = (\text{Measured Value} - \text{Nominal Value}) / \sigma$$

and, the Evidence that a measured value is different from Nominal is given by the formula

$$\text{EVIDENCE} = (\pi / \sqrt{3}) * (\text{Measured Value} - \text{Nominal Value}) / \sigma \qquad (10)$$

In quality control it has become customary to conclude that a process is out of control whenever the measured variable deviates 3 Sigmas from the nominal. This amounts to having $(\pi / \sqrt{3}) * 3$ or 5.44 units of evidence.

6 sigma = 10.88 units of Evidence

APPLYING THE CONCEPT OF EVIDENCE TO LIFE TESTS

In the case of life testing with Weibull Analysis we define

NOMINAL VALUE = ln (Required B-Q Life) = ln (B-Q Goal)

At any B-Q life level the standard error of ln(B-Q) is

$$\sigma_{\ln B_Q} = 1 / ((b * SQR(.5*N*(1 + Q))) \qquad (11)$$

where b = Weibull Slope

N = Sample Size at B-Q Life

Q = Quantile Level

NOTE: This standard error of the natural logarithm of B-Q life is based on the SEMI-PARAMWETRIC approach for CONFIDENCE BAND construction on a Weibull plot. We take the natural logarithm of the B-Q life because Weibull probability paper has a logarithmic life scale for its abscissa.

Thus, if the warranty B-Q life is $_{o}B_Q$ then

$$Z = \left[\ln (\text{Test } B_Q) - \ln (\text{Goal } {}_{o}B_Q) \right] / \sigma_{\ln B_Q} \qquad (12)$$

or, $$Z = b * SQR(.5*N*(1 + Q)) * \ln (\text{Test B-Q} / \text{Goal B-Q}) \qquad (13)$$

Then, the EVIDENCE of compliance to the Goal B-Q is

$$\text{EVIDENCE} = (\pi / \sqrt{3})*Z =$$

$$= (\pi / \sqrt{3})*b * SQR(.5*N*(1 + Q)) * \ln (\text{Test B-Q} / \text{Goal B-Q}) \qquad (14)$$

NOTE: In an acceptable product Test B-Q is greater than Goal B-Q.

When a TEST ENTROPY is being compared to a GOAL ENTROPY at a specified required life goal the Evidence formula becomes

$$\text{EVIDENCE} = (\pi / \sqrt{3})*SQR(.5*N*(1 + Q))*\ln(\text{Goal Entropy} / \text{Test Entropy}) \qquad (15)$$

NOTE: In an acceptable product the Test Entropy is less than the Goal Entropy.

The b factor (Weibull slope) is unity in Entropy comparisons because life has an exponential distribution with respect to Entropy.

TESTING TO DETERMINE EVIDENCE OF COMPLIANCE

Suppose the machine manufacturer we mentioned earlier tests 10 machines to failure and gets the following results in numerical order of hours to failure:

Failure No.	Hours to Failure
1	3858
2	4016
3	4197
4	4374
5	4544
6	4728
7	4906
8	5080
9	5240
10	5420

The Weibull plot for this data set is shown in FIGURE 1. From the statistical program "LEASQ" (in G-W Basic) we obtain the Weibull parameters: b = Weibull Slope = 9.54

θ = Characteristic Life = 4869 hrs.

Now, if the manufacturer is going to sell 1000 machines and have none of them fail in 2000 hours of use, then there must not be any failure in 2000 hours until the next manufactured machine (no. 1001) is used. This means that the MAXIMUM TOLERABLE fraction failed in the warranty period of 2000 hours is 1/ 1001. Then the required RELIABILITY must be at least 1000 / 1001, which requires the MAXIMUM PERMISSIBLE ENTROPY to be – ln(1000 / 1001), which has the numerical value .0009995 as the GOAL ENTROPY to which compliance is required.

Going to the Weibull plot in FIGURE 1 we see that at the promised warranty life of 2000 hours the fraction failed is 1 - EXP(- (2000 / 4869)^9.54) = .000205876, which means that at 2000 hours the TEST RELIABILITY is 1 - .000205876 = .999794124. Thus, TEST ENTROPY on the data plot is -ln(.999794124)=.000205897.

We can now evaluate the TEST EVIDENCE of compliance to a promised service life of 2000 hours by evaluating equation (15):

TEST EVIDENCE

= ($\pi / \sqrt{3}$)*SQR(.5*10*(1 + .000205876))*ln(.0009995 /.000205897)

= 6.41 units of evidence.

For a machine costing $4500 to produce and then sold for $5000 we want to gain at least TWICE as much as we would lose by having to replace any failed machine with a loss of $4500 - $500 = $4000 per replacement. Thus, G = $5000 - $4500 = $500

L = $4500 - $500 = $4000

K = 2 = Desired Profitability Ratio

According to Equation (8):

REQUIRED EVIDENCE of COMPLIANCE = ln (K*L / G) = ln(2*4000 /500)

= ln 16 = 2.77 units of evidence.

Since the TEST EVIDENCE of 6.41 units exceeds the REQUIRED EVIDENCE of 2.77 units, we conclude that these machines are sufficiently compliant to the promised service life of 2000 hours to easily give the manufacturer his desired profitability ratio.

CONCLUSION

From the example discussed in this bulletin we have shown how useful the statistical concepts of ENTROPY and EVIDENCE are in compliance questions on designs with required durability levels in service usage by customers.

FIGURE 1.

WEIBULL PLOT OF THE 10 TESTED MACHINES