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PREDICTING THE REMAINING LIFE RELIABILITY OF AN ITEM WHICH HAS SURVIVED TO A SPECIFIC AGE

INTRODUCTION

Any population of varying lives, be it of biological items or mechanical items, consists of life quantiles, i.e., fractions failed, between 0 and 1. For example, a specific automotive component, such as a bearing type, will behave in such a manner as to have a unique $B_{10\%}$ life, where 10% of the population of such bearings will have failed in service in a given type of usage. Likewise, there is a $B_{20\%}$ life at which 20% of such bearings will have failed in that same type of usage.

Now, we raise the question as to the survival reliability we can have in predicting that a bearing which survived to $B_{10\%}$ life will survive to the $B_{20\%}$ life. The answer is quite simple when we realize that an item which has survived the $B_{10\%}$ life level is at quantile Q = 0.1 and still has a range of possible failure locations between Q = 0.1 and Q = 1.0. Since the difference between Q = 0.1 (at the $B_{10\%}$ life) and Q = 0.2 (at the $B_{20\%}$) is 0.2 - 0.1 = 0.1, and the total difference between the 100th percentile (Q = 1.0) and the $B_{10\%}$ level (Q = 0.1) is 1.0 - 0.1 = 0.9, it follows that the fraction of $B_{10\%}$ life survivors which will fail by the $B_{20\%}$ life is 0.1/0.9 = 0.111111, and, consequently the survival reliability of a $B_{10\%}$ survivor being able to reach the $B_{20\%}$ life level is 1.0 - 0.11111 = 0.88889. Likewise, the survival reliability of a survivor to $B_{10\%}$ being able to survive to $B_{30\%}$ is 1.0 - (0.3 - 0.1)/(1.0 - 0.1) = 1 - 2/9 = 7/9 = 0.77778.

We can construct a table of survival reliabilities for an item which is still unfailed at $B_{10\%}$ life when we ask for its survival reliability to higher quantile levels. Table I on Page 2 is such a table.

TABLE I: (FOR AN ITEM UNFAILED AT THE B10% LIFE)

For Survival To	**Survival Reliability**
$B_{20\%}$ Life	8/9 = 88.89%
$B_{30\%}$ Life	7/9 = 77.78%
$B_{40\%}$ Life	6/9 = 66.67%
$B_{50\%}$ Life	5/9 = 55.56%
$B_{60\%}$ Life	4/9 = 44.44%
$B_{70\%}$ Life	3/9 = 33.33%
$B_{80\%}$ Life	2/9 = 22.22%
$B_{90\%}$ Life	1/9 = 11.11%

By following the same logic as employed in the construction of Table I we shall generalize and come up with a comprehensive survival prediction method of reliability for any specified additional amount of life after an item has already survived to any specific age.

THE MATHEMATICAL GENERALIZATION OF THE SURVIVAL RELIABILITY QUESTION

Let Q_1 = The Quantile Level to which an item has survived
Let Q_2 = The Quantile Level to which we want to predict additional survival reliability

Then, the Additional Life (From Q_1 to Q_2) is

$$x_{Add} = B_{Q2} - B_{Q1} = x_2 - x_1 \quad ,$$

where, B_{Q1} = The Life to Quantile Q_1 = x_1
B_{Q2} = The Life to Quantile Q_2 = x_2

Then, we can write the general formula for the survival reliability from B_{Q1} to B_{Q2} after surviving to B_{Q1} as

$$R_{Add} = R(x_{Add}) = (1 - Q_2)/(1 - Q_1) \tag{1}$$

PREDICTING THE ADDITIONAL LIFE OF SURVIVORS AT ANY AGE

The prediction of the additional amount of life to be expected beyond any attained age is of utmost importance in the life insurance business. Likewise, it is also of utmost importance in reliability questions about consumer products which must comply with guaranteed service periods and durability requirements.

Suppose, for example, that a mechanical part from a Weibull Life Distribution has survived to age x1. How much more life beyond x1 can be expected with reliability Radd? This amounts to using equation (1) on Page 3 for a specified value of the additional survival reliability R_{Add}.

Thus, from $R_{add} = (1 - Q_2)/(1 - Q_1)$ we obtain

$$1 - Q_2 = R_{Add}(1 - Q_1)$$

or

$$R_2 = R_{Add}R_1 \qquad (2)$$

where R_1 = Reliability to age x_1
R_2 = Reliability to age x_2
and R_{Add} = Specified Additional Survival Reliability desired to age x_2 if unfailed at age x_1

It is interesting to note that equation (2) simply expresses the fact that the survival probability from age 0 to age x_2 is the product of the survival probability from age 0 to age x_1 multiplied by the survival probability from age x_1 to age x_2, as shown schematically in Figure 1 below.

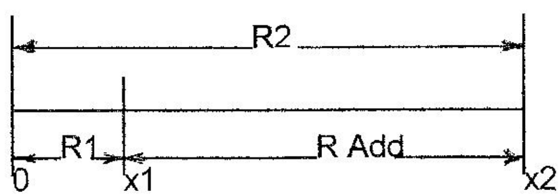

Figure 1

DETERMINING THE ADDITIONAL LIFE BEYOND ANY AGE BY USING ENTROPY

From equation (2) on Page 4: $1/R_2 = (1/R_{Add})\cdot(1/R_1)$

So, taking natural logarithms: $\ln(1/R_2) = \ln(1/R_{Add}) + \ln(1/R_1)$ (3)

Now, by definition: $\ln(1/R_2)$ = Entropy at $x_2 = \varepsilon_2$

$\ln(1/R_{Add})$ = Additional Entropy $= \varepsilon_{Add}$
(From age x_1 to age x_2)

$\ln(1/R_1)$ = Entropy at $x_1 = \varepsilon_1$

So, we can write equation (3) as

$$\varepsilon_2 = \varepsilon_{Add} + \varepsilon_1 \qquad (4)$$

Thus, $\varepsilon_{Add} = \varepsilon_2 - \varepsilon_1 = \ln(1/R_2) - \ln(1/R_1)$

Thus, the additional amount of entropy beyond life x_1 with reliability R_{Add} is $\ln(1/R_{Add})$.

In a Weibull Distribution of slope b and characteristic life θ, the entropy at any life is $(x/\theta)^b$.

From (4):
$$\left(\frac{x_2}{\theta}\right)^b = \left(\frac{x_1}{\theta}\right)^b + \ln\left(\frac{1}{R_{Add}}\right)$$

So,
$$x_2 = \theta\left[\left(\frac{x_1}{\theta}\right)^b + \ln\left(\frac{1}{R_{Add}}\right)\right]^{\frac{1}{b}}$$

and
$$x_{Add} = x_2 - x_1 = \theta\left[\left(\frac{x_1}{\theta}\right)^b + \ln\left(\frac{1}{R_{Add}}\right)\right]^{\frac{1}{b}} - x_1 \qquad (5)$$

A NUMERICAL EXAMPLE

Suppose an item comes from a Weibull Population with slope 2 and a characteristic life of 1,000 hours.

Question 1: If the item has survived to 2 00 hours, how many more hours can be expected beyond the 200 hours with a survival reliability $R_{Add} = 0.9$ for the additional hours beyond the initial 200 hours?

Solution 1:

In this case: $x_1 = 200$

$b = 2$

$\theta = 1000$

So, $$X_{Add} = x_2 - x_1 = 1000[(200/1000)^2 + \ln(1/.9)]^{1/2} - 200$$
$$= 181.26 \text{ additional hours with 90\% reliability}$$

Question 2: If we only require 50% reliability for the additional life beyond the first 200 hours, how many more hours can we go beyond the 200 hours already survived?

Solution 2:

Answer: $$X_{Add} = 100[(200/1000)^2 + \ln(1/.5)]^{1/2} - 200$$
$$= 656.24 \text{ additional hours with 50\% reliability}$$

Thus, it can be seen that higher reliabilities for additional hours of survival permit fewer hours for the promised additional life.

CONCLUSION

Once more, as in other questions of reliability, the concept of **Entropy** has turned out to be a most useful tool in predicting how much more life can be expected of an item which has survived to some specific age. The whole trick simply amounts to adding on the value $\ln(1/R_{Add})$ to the entropy already reached at the item's present age. This sum is then the final entropy at failure with the specified reliability R_{Add} for the additional life.