STATISTICAL BULLETIN
Reliability & Variation Research

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Volume 28
Bulletin 3

July, 1998
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A THEORETICAL CHECK ON THE RESULTANT ODDS CONCEPT FOR THE SYSTEM RELIABILITY RESULTING FROM THE MULTIPLICATION OF TWO INDEPENDENT COMPONENT RELIABILITIES IN SERIES WHEN THE CONFIDENCE LEVEL FOR EACH OF THEM IS GIVEN

INTRODUCTION

In the field of statistical analysis of reliability data one of the most useful concepts we can employ in such analysis is the concept of order statistics. By definition, order statistics are simply collections of numbers representing some type of measured variable with the values of the measurement placed in numerical order from the smallest value to the largest value.

In dealing with the reliability of systems we are faced with the question of confidence levels for system reliabilities when the individual component reliabilities each have their own given confidence levels. In other words, we must answer such questions as "How extreme is the product of two extreme values from two samples of order statistics of given sizes?". Or, we might ask "How does the product of the two middle values in two sets of order statistics turn out to be with respect to the set of ordered values in the list of all possible combinations from two samples of order statistics?". Then, in general, we can ask for the ordered location of a product of two values from a couple of samples of ordered measurements, such as reliabilities, when we take the entire collection of products in the larger sample size equal to the product of the two individual sample sizes.

In this bulletin we shall answer all these questions and in that fashion come up with a mathematical justification for the multiplication of odds when we combine component reliabilities by the product rule for systems of components in series.

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FUNDAMENTALS OF THE MEAN RANKING OF ORDER STATISTICS

If we have a sample of N_1 order statistics and another sample of N_2 order statistics and we form all possible products from the multiplication of some value in the first sample by some value in the second sample, we will end up with $N_1 N_2$ possible products by taking all the pairs of factors which can be formed from the two samples.

If we take order statistic number J_1 from the first sample of N_1 then there will be a mean confidence of $J_1/(N_1 + 1)$ of going below J_1. Likewise, if we take order statistic number J_2 from the second sample of size N_2 there will be a mean confidence of $J_2/(N_2 + 1)$ of going below J_2. Then by superimposing these two confidences, i.e., $C_1 = J_1/(N_1 + 1)$ an $C_2 = J_2/(N_2 + 1)$ we obtain the resultant confidence $C_r = C_1C_2/[C_1C_2 + (1 - C_1)(1 - C_2)]$.
Substituting the values of C_1 and C_2 in terms of J_1, J_2, N_1, and N_2, We obtain the formula for the resultant confidence as

$$C_r = \frac{J_1 J_2}{J_1 J_2 + (N_1 + 1 - J_1)(N_2 + 1 - J_2)}$$

Now, we ask what order statistic number in sample size N_1N_2 (the product of the two sample sizes) will yield this resultant confidence? Let x = the order statistic number we are seeking in a sample of size N_1N_2 which will yield the same resultant mean rank confidence.

Then,
$$\frac{x}{(N_1 N_2 + 1)} = \frac{J_1 J_2}{J_1 J_2 + (N_1 + 1 - J_1)(N_2 + 1 - J_2)}$$

so,
$$x = \frac{J_1 J_2 (N_1 N_2 + 1)}{J_1 J_2 + (N_1 + 1 - J_1)(N_2 + 1 - J_2)}$$

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NUMERICAL EXAMPLES OF THE THEORY OF MULTIPLIED ORDER STATISTICS

EXAMPLE #1

#2 in 5 multiplied by #3 in 10 becomes #x in 50, where

$$x = \frac{(2)(3)(51)}{(2)(3) + (4)(8)} = \frac{6(51)}{32 + 6} = \frac{306}{38} = 8.05263$$

So, on the average,
the 2nd in 5 multiplied by the 3rd in 10 becomes #8.05263 in 50.

EXAMPLE #2

In this example we show that two multiplied extreme values go into the extreme value of their product. For example, take #1 in 6 and multiply it by #1 in 7. According to the theoretical formula, such a product will become #x in 42, where

$$x = \frac{(1)(1)(43)}{(1)(1) + (6)(7)} = \frac{43}{43} = 1$$

EXAMPLE #3

In this example we show how two multiplied middle values go into the middle value of the set of products.

Take #3 in 5 and multiply it by #4 in 7. This becomes #x in 35, where

$$x = \frac{(3)(4)(36)}{(3)(4) + (3)(4)} = \frac{(12)(36)}{24} = 18$$

Thus, the 3rd in 5 multiplied by the 4th in 7 becomes #18 in 35. Note that #18 in 35 is the middle value for a sample of 35.

CONCLUSION

Eureka! We have found the secret of multiplying order statistics and how their products fit into the sample size equal to the product of the two individual sample sizes. This is exactly in agreement with the multiplication of odds for the resultant odds on the system reliability obtained by multiplying two independent component reliabilities!