STATISTICAL BULLETIN
Reliability & Variation Research

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An Introduction to Reliability and Entropy

Introduction

The most fundamental concept involved in a failure process (and, hence, in survival reliability) is the concept of **Entropy**. Because of the great importance of this concept in the understanding of the statistics of life survival of consumer products, we have decided to devote this particular bulletin to the basic concepts and their definitions as an introduction to the general field of **Reliability** and **Entropy.** Once these basic ideas have been introduced we can devote subsequent bulletins to present more advanced ideas which are the latest ones used in our most up-to-date techniques for our present day courses in **New and Effective Methods** of tackling problems of life testing programs and product reliability.

The Universal Cumulative Distribution Function of Life

A cumulative distribution function of life is a function whose independent variable [abscissa **x**] is Life, and whose dependent variable [ordinate **F(x)**] is the fraction of the population failed in any time x.

Note: We abbreviate cumulative distribution function as **cdf**

We use the capital letter **F** to denote a cumulative distribution function. Thus, **F(x)** represents the fraction failed in time x. The life x (i.e., service time) ranges from some **minimum life**, where $F(x) = 0$, to some **maximum life**, where $F(x) = 1$.

The minimum life could be **zero** or some value **greater than zero** and the maximum life could be finite or infinite.
(Note: In the real world maximum life is most likely finite. Infinite life is an idealistic assumption.)

In natural failure phenomena, i.e., in situations involving deterioration, break-down, weakening by use, wear-out, fatigue, etc., we speak of the concept of **Entropy.** It is the absolute value of the natural **logarithm of a probability of success.** Success in service life is what we call survival, and the **probability of survival** to any service life x is called the **reliability to target x.**

Reliability to target x is denoted by the function R(x). By the definition of Entropy we conclude that

$$\text{Entropy to Target } x \;=\; \text{Absolute Value of } \ln [R(x)].$$

Since R(x) is a probability (of success), it follows that its logarithm is negative (since probabilities are always between 0 and 1). Therefore, if we denote Entropy at x by the symbol $\boldsymbol{\varepsilon}(x)$ we can write

$$\text{Entropy at } x \;=\; \boldsymbol{\varepsilon}(x) \;=\; -\ln [R(x)] \,.$$

Since $\boldsymbol{\varepsilon}(x) = -\ln [(x)]$, it follows that

$$\ln [R(x)] \;=\; -\,\boldsymbol{\varepsilon}(x) \quad \text{or} \quad R(x) \;=\; \exp [-\boldsymbol{\varepsilon}(x)] \,.$$

Since $R(x) = \exp [-\boldsymbol{\varepsilon}(x)]$ (the survival probability to x), it follows that the failure probability to x is

$$F(x) \;=\; 1 \;-\; R(x) \;=\; 1 \;-\; \exp [-\boldsymbol{\varepsilon}(x)] \,.$$

Thus, we have derived the **Universal Cumulative Distribution Function of Life** in terms of Entropy. The basic failure law of nature is

$$\text{Failure Probability} \;=\; 1 \;-\; \exp (-\text{Entropy}) \,.$$

Note: Failure probability expressed as a percent is the expected percent failed of the population to time x .

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The Probability Density Function of Life

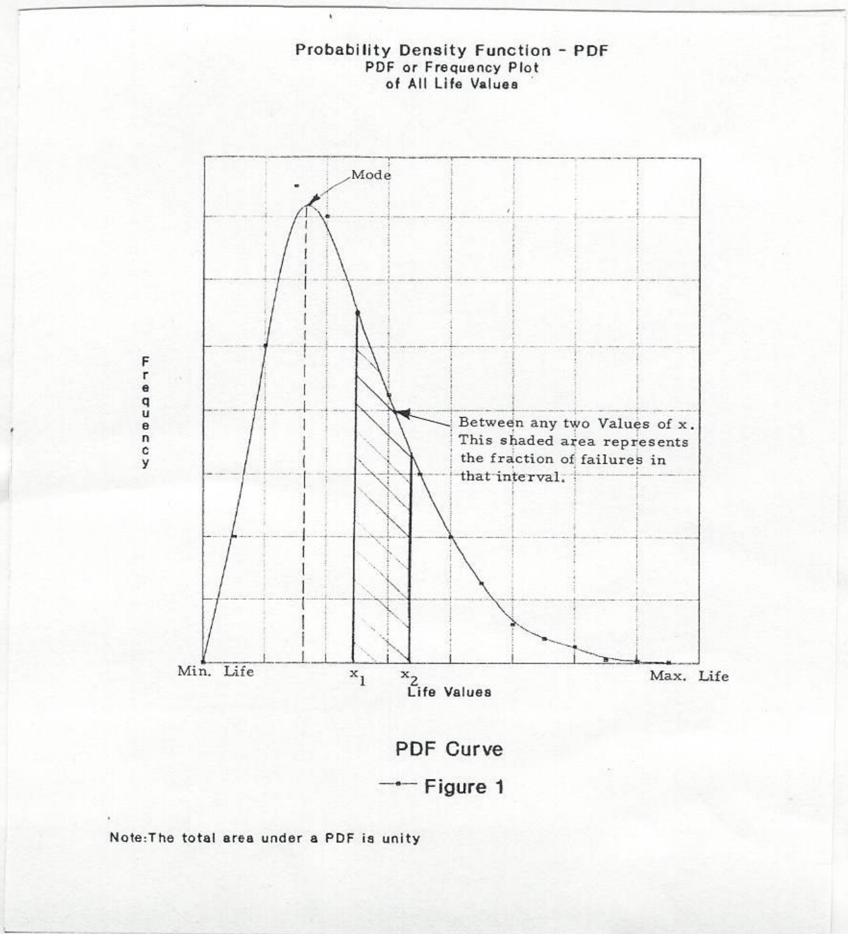
A Probability Density Function (i.e., a frequency function), for a failure phenomenon possessing a range of different times to failure (such as a Normal Distribution) is obtained by taking the first derivative of the process cumulative distribution of life. Thus,

if F(x) = cumulative distribution function

then d[F(x)]/dx = f(x) = probability density function .

Note: Small **f** is used for the probability density function. We abbreviate probability density function as **pdf .**

The probability density function turns out to be a frequency plot of failure occurrences at different points of the whole range of possible life values.

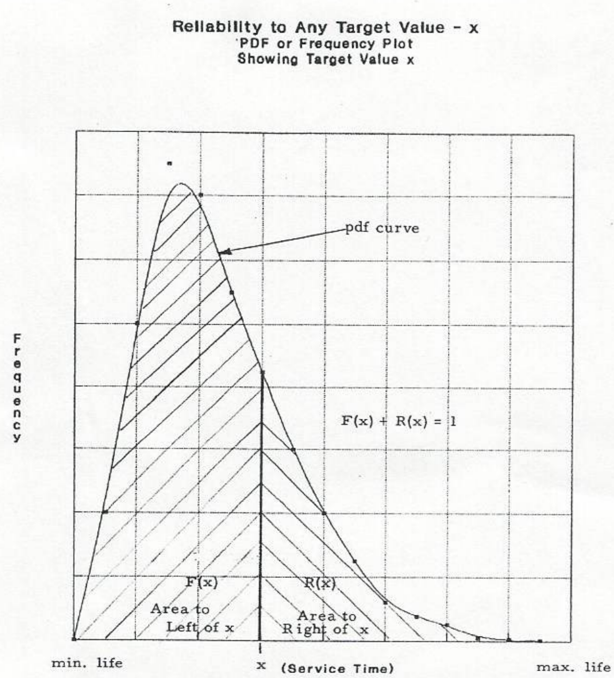


From the universal cdf of life, i.e., $F(x) = 1 - \exp[-\varepsilon(x)]$, we obtain the pdf

$$f(x) = \frac{d[\varepsilon(x)]}{dx} \cdot \exp[-\varepsilon(x)]$$

where $d[\varepsilon(x)]/dx$ is the derivative of the Entropy function $\varepsilon(x)$ with respect to x. It is also known as the **Hazard Function.**

The term $\exp[-\varepsilon(x)]$ is the **Universal Reliability Function.** Thus, the probability density function is the product of the **Hazard function** and the **Universal Reliability function.** The service time x at which the failure (as given by the pdf) is a maximum is called the **Mode** of the distribution (reference Figure 1). From a picture of a pdf as shown below, we can indicate the values of the cdf, F(x), as well as the reliability R(x) to any target x.

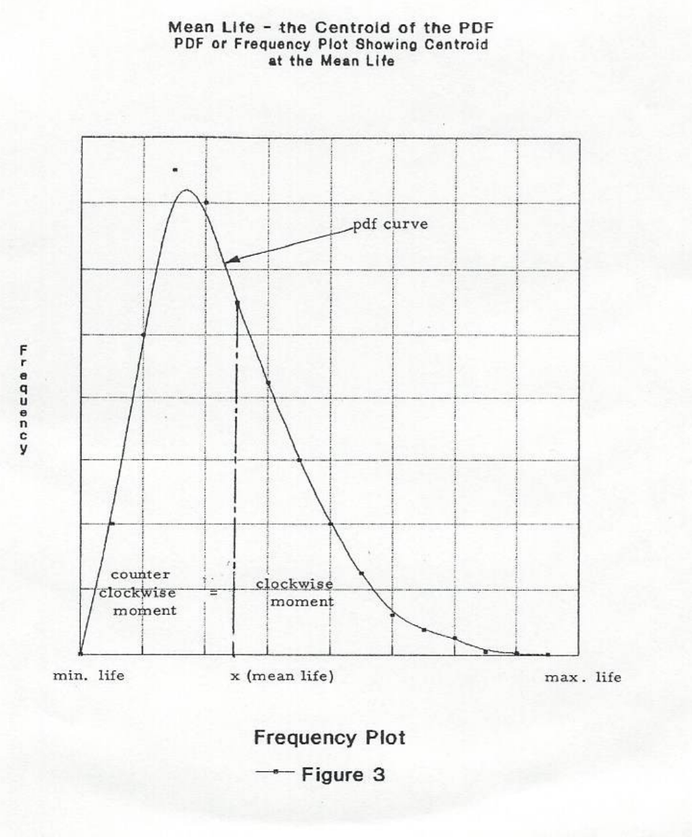


Frequency Plot

Figure 2

The **median life** is the value of x at which F(x) = 0.5, i.e., the x at which 50% of the entire unit area under the pdf curve is to the left of x, (reference to Figure 2). The B_{10} life is the x value which has 10% of the entire area to its left i.e., F(x) = 0.10 . The B_q life is the x which has the fraction q of the entire area to its left.

The **mean life** is the point on the x axis at which the pdf will balance, i.e., the x at which clockwise and counter clockwise moments are equal. This is known as the centroid of the pdf's area and is shown graphically in Figure 3.



The Hazard Plot and Cumulative Hazard - (Entropy)

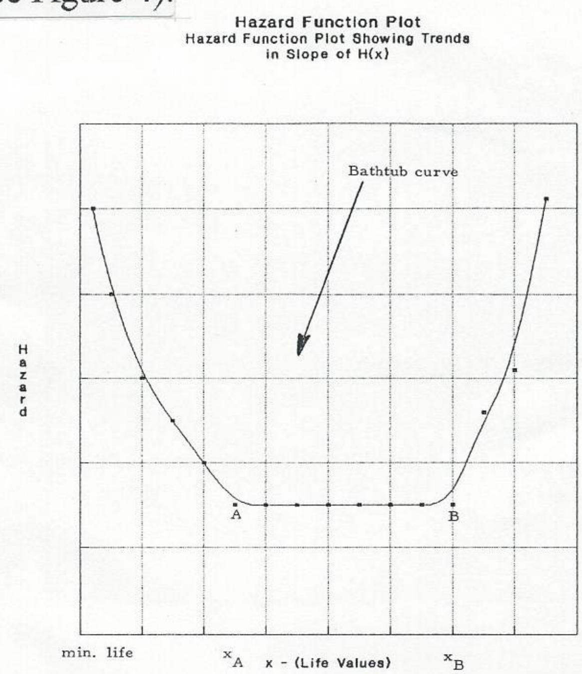
The Hazard function is denoted by the notation H(x). By definition the Hazard function is the derivative of Entropy.

$$H(x) = d[\varepsilon(x)]/dx$$

The inverse of a derivative is an integral. Thus, the Entropy is equal to the integral of Hazard.

$$\varepsilon(x) = \int_0^x H(x)\,dx$$

The integral of Hazard is known as the **Cumulative Hazard** . Thus, Entropy and Cumulative Hazard are identical. Graphically, the Hazard function typically plots as a bathtub curve, or one or more portions of such a curve, (Reference Figure 4).



Hazard Plot

Figure 4

Since Entropy is the integral of Hazard it follows that the Entropy at x is equal to the area under the bathtub curve from minimum life to time x .

$$\varepsilon(x) = \int_{\alpha}^{x} H(x)\,dx$$ <u>Note:</u> $\alpha = 0$ or min. life

The three distinct portions of a Hazard curve (i.e., bathtub curve) are: (Reference Figure 4)

a. The decreasing portion (left of point A)

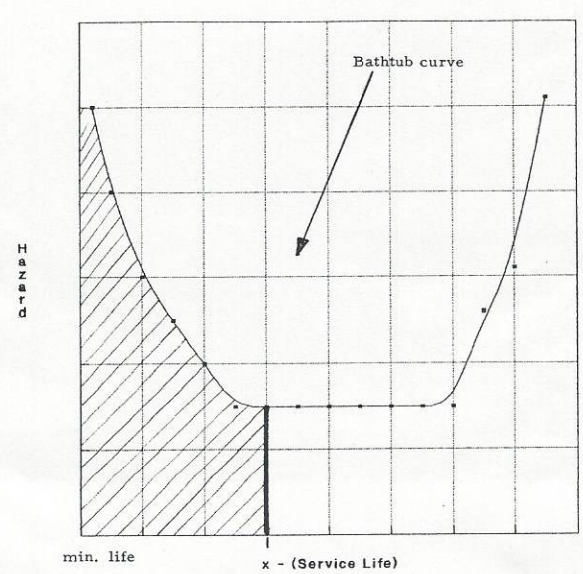
b. The constant portion (between point A and point B)

c. The increasing portion (right of point B)

Note: In special cases, a Hazard curve might have only one or two of these portions.

The Entropy at time x, (i.e., service life) is represented graphically as follows:

Entropy(Area Under Hazard Function Plot)
Hazard Function Plot Showing
Entropy as Area Under Curve



Entropy Under H(x)

Figure 5

A table of Entropy vs. Percent Failed for first failures can be constructed showing their one to one correspondence. Recall that the failure probability is

$$F(x) = 1 - R(x) = 1 - \exp[-\varepsilon(x)] .$$

Percent Failed - F(x)	Entropy - $\varepsilon(x)$
1	.01005
5	.05129
10	.10536
20	.22314
30	.35667
40	.51038
50	.69315
60	.91629
70	1.20397
80	1.60944
90	2.30259
95	2.99573
99	4.60517

Increased Entropy means more failures (i.e., a higher percent failed) in a population. In a repairable system we can state that Entropy is equal to failures per system (i.e., the number of break-downs) . Entropy always increases with age, (age = service time) .

Special Case of the Weibull Distribution

Back in the 1940's and 1950's a Swedish engineer by the name of Weibull suggested failure distributions of mechanical parts might be represented by using a simple power function of service time as the Entropy formula. Thus, Weibull suggested that we take

$$\varepsilon(x) = (x/\theta)^b = \textbf{Entropy at x,}$$

i.e., the b^{th} power of x multiplied by a constant. The multiplying constant in this formula is $\mathbf{1/(\theta)^b}$. The reason for using this form of multiplying constant is to keep Entropy **dimensionless**, as it should be.

In Weibull's Entropy formula the constant

b - is called the **Weibull Slope**
θ - is called the **Characteristic Life** .

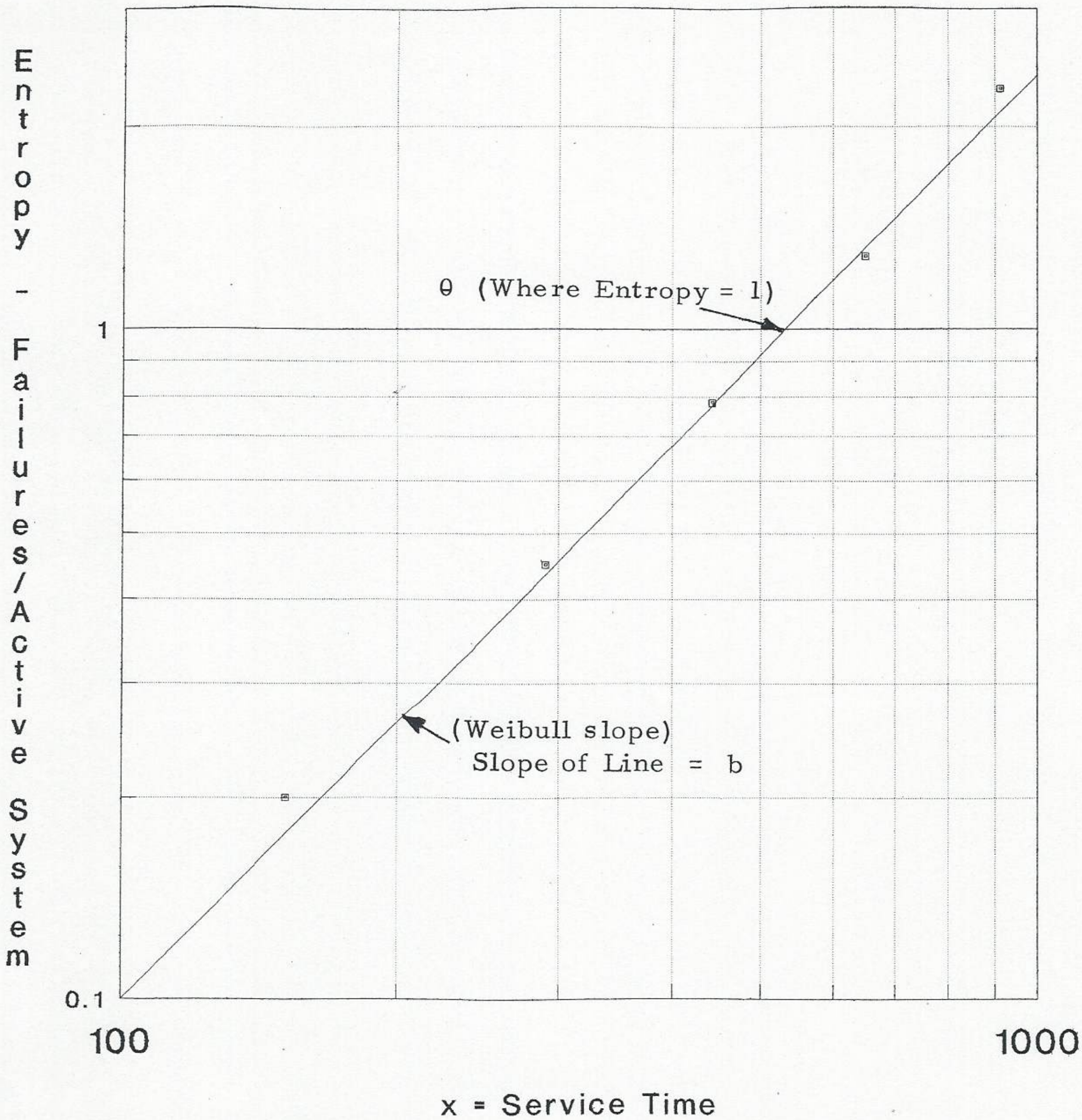
It can be seen that at the Characteristic Life the Entropy is 1.0 or **unity.** Taking the natural logarithm of both sides of the equation $\varepsilon(x) = (x/\theta)^b$ yields

$$\ln \varepsilon(x) = b \ln x - b \ln \theta$$

Thus it can be seen that in the Weibull Entropy function there is a linear relation between **ln (Entropy)** and **ln (Service Time).** Therefore, in case of a Weibull failure phenomenon we get a straight line on log-log paper with life plotted horizontally and Entropy plotted vertically.

Remember: For repairable systems Entropy is the **number of failures per active system.**

A Weibull phenomenon with parameters **b** and θ will look like that shown in Figure 6 when using log-log paper.



Log-Log Weibull Plot

FIGURE 6

In the formula $\varepsilon(x) = (x/\theta)^b$ the Minimum Life is zero, i.e., where Entropy is equal to zero. Also, the Maximum Life is infinity, i.e., where Entropy is equal to infinity. Weibull also permitted the minimum life to be some positive time $x = \alpha$. In that case he proposed the Entropy function

$$\varepsilon(x) = [\ (x - \alpha) / (\theta - \alpha)\ ]^b$$

is the **Three-Parameter Weibull** where

α = Minimum Life
θ = Characteristic Life
b = Weibull Slope (power function exponent)

Note: In order to make a linear plot on log-log paper from the Three-Parameter Weibull Entropy function we must plot **$(x - \alpha)$** instead of x on the horizontal axis.

The Two-Parameter Weibull cdf is given by

$$F(x) = 1 - \exp[-\varepsilon(x)] .$$

Therefore,

$$F(x) = 1 - \exp[-(x/\theta)^b] .$$

The Three-Parameter Weibull cdf is given by

$$F(x) = 1 - \exp[-\varepsilon(x)] .$$

Therefore,

$$F(x) = 1 - \exp\{-[(x - \alpha) /(\theta - \alpha)]^b\} .$$