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ODDS AND THEIR CHANCES OF BEING EXCEEDED

INTRODUCTION

Every statistical parameter estimated from a data sample is subject to error and uncertainty. This is because a sample cannot precisely tell us the true nature of a population, but is subject to error, since it only represents partial evidence about the nature of the whole population. For example, the average weight of five randomly selected males belonging to a men's club with hundreds of members is not necessarily the true average for the entire club. The same is the case for any other parameter, such as the standard deviation in normal analysis or the Weibull slope in Weibull analysis of sample data. Not only is this type of sampling uncertainty present in means, standard deviations, and Weibull slopes, but also in ODDS calculated from samples collected in an attempt to show that a new product is superior to some standard target or previous design .

In this bulletin we shall derive the two extremes of ODDS , going from the lowest (most conservative) odds to the highest (most optimistic) odds, and then constructing an EXCEEDANCE PROBABILITY DIAGRAM of the Odds Ratio for the total situation in any investigation based on specific sample sizes.
In essence, we shall show the Stochastic nature of odds estimates .

THE UNIVERSAL LAW OF ODDS

Just as in other branches of science there are basic laws or rules to guide us in analyzing the quantitative aspects of a problem, so in decision statistics we have a guiding principle known as the UNIVERSAL LAW OF ODDS , which is expressed mathematically by the formula

$$\text{ODDS} = (\text{ENTROPY RATIO})^{(\text{ODDS EXPONENT})} \quad . \qquad \text{(I)}$$

In order to use Formula (I) in a decision about the odds in favor of a real difference between two estimated distribution functions from separate populations at a particular value of the independent variable measured in each population (say, at 1000 hours of life), we note the Entropy level of each distribution at 1000 hours (or at any desired life goal) and from these Entropy values we calculate the ratio (Larger Entropy/Smaller Entropy).

The Entropy at any life x in a cumulative distribution function $F(x)$ is defined as follows :

$$\text{ENTROPY at } x = = \mathcal{E}(x) = \ln \frac{1}{1 - F(x)} \quad , \quad \text{where}$$

$F(x)$ = fraction failed at life x . Hence, if we have two estimated population cumulative distribution functions $F_1(x)$ and $F_2(x)$, where $F_1(x) > F_2(x)$, we would calculate

$$\text{ENTROPY RATIO} = \frac{\ln\left[\dfrac{1}{1 - F_1(x)}\right]}{\ln\left[\dfrac{1}{1 - F_2(x)}\right]} \quad .$$

The ODDS EXPONENT would have two extreme values :

$$\text{OPTIMISTIC VALUE (i.e., highest value)} = \frac{\sqrt{1 + \dfrac{\sqrt{N_1 N_2}}{\frac{1}{2}(N_1 + N_2)}}}{.55\left(1/\sqrt{N_1} + 1/\sqrt{N_2}\right)}$$

N_1 = size of sample #1 having CDF $F_1(x)$

$$\text{CONSERVATIVE VALUE (i.e., lowest value)} = \frac{\sqrt{1 + \dfrac{\sqrt{N_1 N_2}}{1/2(N_1 + N_2)}}}{.55\left(1/\sqrt{N_1 F_1(x)} + 1/\sqrt{N_2 F_2(x)}\right)}$$

N_2 = size of sample #2 having CDF $F_2(x)$

ILLUSTRATIVE EXAMPLE OF OPTIMISTIC ODDS AND CONSERVATIVE ODDS

Let us suppose we have an old design of which we tested $N_1 = 5$ items and obtained Weibull parameters b_1 = Weibull slope = 2

θ_1 = Characteristic life = 1600 hours.

This would give the old design a CDF

$$F_1(x) = 1 - e^{-(x/1600)^2} .$$

Next, suppose a new design is tested with $N_2 = 8$ items , giving Weibull parameters b_2 = Weibull slope = 2.4

θ_2 = Characteristic life = 2250 hours

and , thus , a CDF

$$F_2(x) = 1 - e^{-(x/2250)^{2.4}} .$$

QUESTION : What are the odds in favor of the new design showing better reliability (fewer population failures) to a target life of 1000 hours ?

SOLUTION

FOR OLD DESIGN : Entropy at 1000 hours $= \ln \dfrac{1}{1 - F_1(1000)}$

$$= (1000/1600)^2 = .390625$$

FOR NEW DESIGN : Entropy at 1000 hours $= \ln \dfrac{1}{1 - F_2(1000)}$

$$= (1000/2250)^{2.4} = .142811$$

ENTROPY RATIO = .390625/.142811 = 2.73526

OPTIMISTIC ODDS EXPONENT

$$= \frac{\sqrt{1 + \dfrac{\sqrt{(5)(8)}}{1/2(5 + 8)}}}{.55\left(1/\sqrt{5} + 1/\sqrt{8}\right)} = 3.189301$$

CONSERVATIVE ODDS EXPONENT

$$= \frac{\sqrt{1 + \dfrac{\sqrt{(5)(8)}}{1/2(5 + 8)}}}{.55\left(1/\sqrt{5}\,F_1(1000) + 1/\sqrt{8}\,F_2(1000)\right)} = 1.454706$$

NOTE : $F_1(1000) = 1 - e^{-(1000/1600)^2} = .323366$

$F_2(1000) = 1 - e^{-(1000/2250)^{2.4}} = .133082$

OPTIMISTIC ODDS $= (\text{ENTROPY RATIO})^{\left(\text{OPTIMISTIC ODDS EXPONENT}\right)} = (2.73526)^{3.189301}$

$= 24.758$

CONSERVATIVE ODDS $= (\text{ENTROPY RATIO})^{\left(\text{CONSERVATIVE ODDS EXPONENT}\right)} = (2.73526)^{1.454706}$

$= 4.322$

THE EXCEEDANCE PROBABILITY DIAGRAM FOR ODDS RATIOS

In the example we calculate the extreme values of estimated odds in favor of the new design being superior to the old design at 1000 hours.

These extremes are :

(1) Lower extreme odds ratio = 4.322 to 1

(2) Upper extreme odds ratio = 24.758 to 1

We define the lower extreme odds ratio as having an exceedance probability of 1 , i.e., a probability of 100% of being exceeded.

Furthermore , we define the upper extreme odds ratio as having an exceedance probability of 0 (zero) , i.e., no chance of being exceeded.

Now we are able to construct the complete EXCEEDANCE PROBABILITY DIAGRAM shown in Figure 1 by plotting the lower extreme odds ratio (4.322) at exceedance probability 1 and the upper extreme odds ratio (24.758) at exceedance probability 0 (zero) . At 50% probability we would use the formula

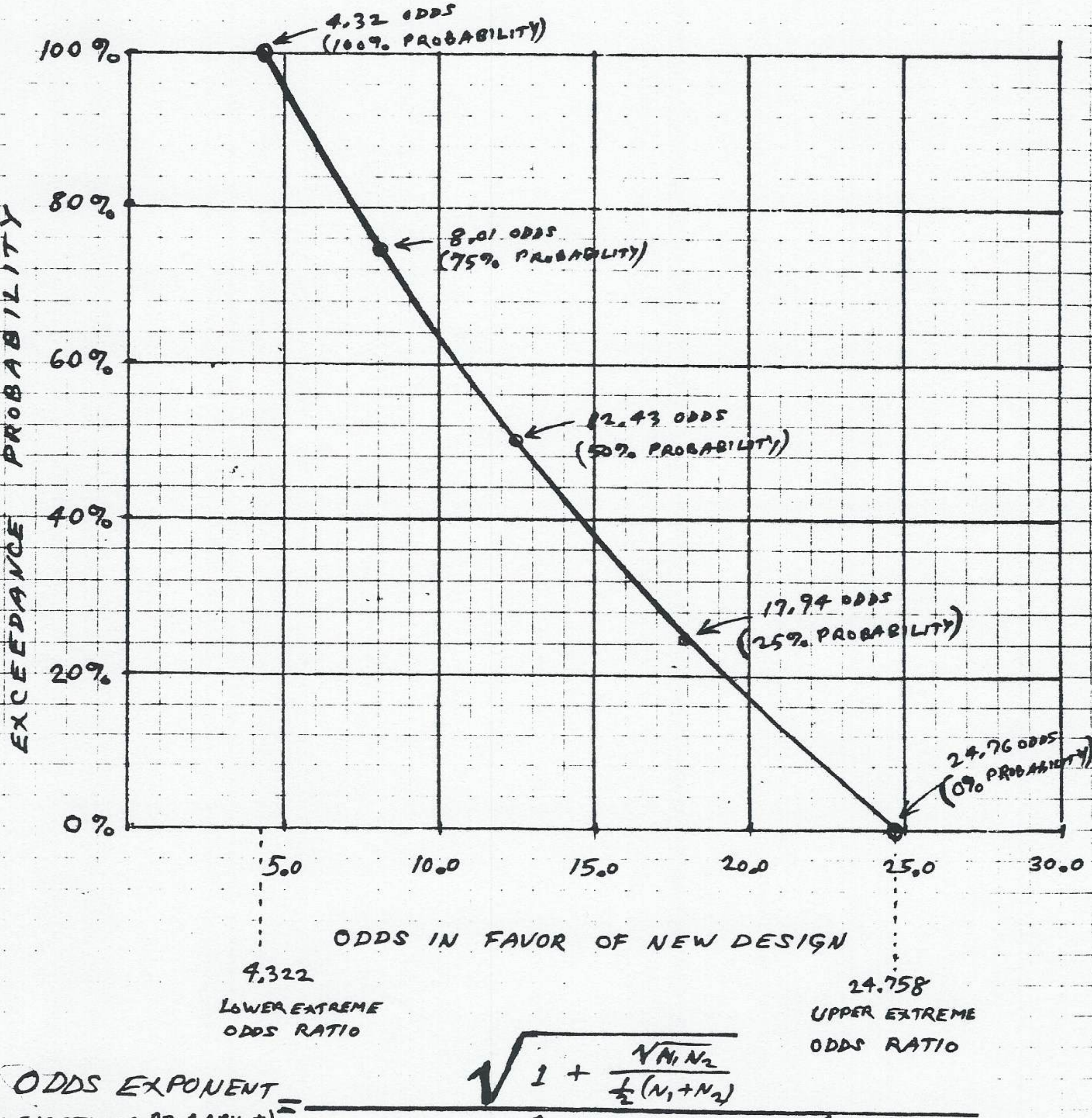
$$\text{MEDIAN ODDS EXPONENT} = \frac{\sqrt{1 + \dfrac{\sqrt{N_1 N_2}}{\frac{1}{2}(N_1 + N_2)}}}{.55\left(\dfrac{1}{\sqrt{N_1\left[.5 + .5\,F_1(x)\right]}} + \dfrac{1}{\sqrt{N_2\left[.5 + .5\,F_2(x)\right]}}\right)}$$

Then ,

$$\text{MEDIAN ODDS} = (\text{ENTROPY RATIO})^{(\text{MEDIAN ODDS EXPONENT})}$$

FIGURE 1

EXCEEDANCE PROBABILITY DIAGRAM FOR ODDS RATIOS
(FOR THE EXAMPLE WITH $N_1 = 5$ AND $N_2 = 8$)



ODDS EXPONENT (FOR EXCEEDANCE PROBABILITY EQUAL TO P)

$$\text{ODDS EXPONENT} = \frac{\sqrt{1 + \dfrac{\sqrt{N_1 N_2}}{\frac{1}{2}(N_1+N_2)}}}{.55\left(\dfrac{1}{\sqrt{N_1\left[1-P+P\,F_1(x)\right]}} + \dfrac{1}{\sqrt{N_2\left[1-P+P\,F_2(x)\right]}}\right)}$$

THEN, ODDS (WITH EXCEEDANCE PROBABILITY P) = (ENTROPY RATIO)^(ODDS EXPONENT)

$$\text{ODDS}\left(\text{with exceedance probability } P\right) = \left(\text{ENTROPY RATIO}\right)^{(\text{ODDS EXPONENT})}$$

CONCLUSION

What we have discussed in this bulletin is just another example of the fact that any statistical estimate is only an estimate which is subject to uncertainty between a lower limit and an upper limit. This universal defect of statistical estimates even applies to estimates of odds by odds makers who derive their figures from sample data. It is just a reminder that sampling data are subject to uncertainty and the amount of uncertainty is a function of sample sizes as well as the amount of prior information we have about population .