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ANALYZING CUMULATIVE DAMAGE BY ENTROPY

INTRODUCTION

The concept of Entropy (i.e., Statistical Weakness) has a multitude of useful applications in the fields of product durability and reliability. One of the most interesting applications of Entropy is in the general treatment of cumulative damage analysis questions. In this bulletin we shall show how this concept of Entropy can be applied to cumulative damage situations. Among other things, we shall show that Miner's Rule is the result of Entropy analysis when the Weibull slope remains fixed for different stress levels under which fatigue life has a two-parameter Weibull distribution. It will, furthermore, be shown how systematically we can handle duty cycles in cases of repeated application of specific stress levels over and over again in a cyclic fashion.

A NUMERICAL EXAMPLE OF A THREE-LEVEL DUTY CYCLE

Suppose an item is subjected to a duty cycle defined as follows :

			Weibull Parameters :
One Round	At $Stress_1$:	5000 cycles	$b_1 = 2.5$; $\theta_1 = 100,000$ cycles
	At $Stress_2$:	3000 cycles	$b_2 = 2.5$; $\theta_2 = 50,000$ cycles
	At $Stress_3$:	2000 cycles	$b_3 = 2.5$; $\theta_3 = 20.000$ cycles

This type of loading with the 3 stresses is repeated round after round until failure occurs.

QUESTION : What is the reliability (i.e., survival probability) of the item to 30,000 cycles of this type of repeated cyclic loading ?

SOLUTION

30,000 cycles represents 3 rounds of the duty cycle, since one round is (5000 + 3000 + 2000) = 10,000 cycles.

The first 5000 cycles under Stress_1 produces an Entropy_1 equal to

$$\mathcal{E}_1(5000) = (5000/100{,}000)^{2.5} .$$

Under Stress_2, this same Entropy would have been produced in 2500 cycles, because $(2500/50{,}000)^{2.5} = (5000/100{,}000)^{2.5}$.

So, after Stress_2 is done, there will be an Entropy equal to what Stress_2 produces in a total of 5500 cycles, i.e.,

$$\mathcal{E}_2(5500) = (5500/50{,}000)^{2.5}$$

Under Stress_3, this same Entropy would have been produced in 2200 cycles,

$$(2200/20{,}000)^{2.5} = (5500/50{,}000)^{2.5} .$$

So, after Stress_3 is done in the first round of the duty cycle there will be an Entropy total equal to what Stress_3 alone produces in a total of 4200 cycles, i.e.,

$$\mathcal{E}_3(4200) = (4200/20{,}000)^{2.5} .$$

So, in one round of 10,000 cycles, i.e., (5000 + 3000 + 2000) cycles, there must be a resultant characteristic life $\hat{\theta}$, such that $(4200/20{,}000)^{2.5} = (10{,}000/\hat{\theta})^{2.5}$, which makes $\hat{\theta} = (20{,}000)(10{,}000)/4200 = 47{,}619$ cycles. The Resultant Entropy at 30,000 cycles of this type of cyclic loading is

$$\hat{\mathcal{E}}(30{,}000) = (30{,}000/47{,}619)^{2.5} = .63$$

Hence the Reliability to 30,000 cycles of this cyclic type of loading is

$$\hat{R}(30{,}000) = e^{-.63} = \underline{.5323} \quad \text{(Ans.)}$$

GENERAL CASE OF A THREE-LEVEL DUTY CYCLE (FOR FIXED WEIBULL SLOPE b)

		Weibull Parameters
One Round	At Stress S_1 : N_1 Cycles	b and θ_1
	At Stress S_2 : N_2 Cycles	b and θ_2
	At Stress S_3 : N_3 Cycles	b and θ_3

$$\mathcal{E}_1 = \mathcal{E}_1(N_1) = (N_1/\theta_1)^b = \left[(\theta_2 N_1/\theta_1)/\theta_2\right]^b$$

$$\mathcal{E}_2(\theta_2 N_1/\theta_1 + N_2) = (N_1/\theta_1 + N_2/\theta_2)^b = \left[\theta_3(N_1/\theta_1 + N_2/\theta_2)/\theta_3\right]^b$$

$$\mathcal{E}_3\left[\theta_3(N_1/\theta_1 + N_2/\theta_2) + N_3\right] = (N_1/\theta_1 + N_2/\theta_2 + N_3/\theta_3)^b =$$

Entropy After (1) Round of Duty Cycle

The Entropy after (1) round of $(N_1 + N_2 + N_3)$ cycles of cyclic loading with a resultant characteristic life of $\hat{\theta}$ is

$$\left[(N_1 + N_2 + N_3)/\hat{\theta}\right]^b .$$

Thus,

$$(N_1/\theta_1 + N_2/\theta_2 + N_3/\theta_3)^b = \left[(N_1+N_2+N_3)/\hat{\theta}\right]^b$$

or,

$$N_1/\theta_1 + N_2/\theta_2 + N_3/\theta_3 = (N_1 + N_2 + N_3)/\hat{\theta} \qquad \text{Miner's Rule}$$

We get the same numerical answer in the first example by applying Miner's Rule, as follows:

$N_1 = 5000$	$N_2 = 3000$	$N_3 = 2000$
$\theta_1 = 100,000$	$\theta_2 = 50,000$	$\theta_3 = 20,000$

Then, by Miner's Rule (derived above), $5000/100,000 + 3000/50,000 + 2000/20,000 = 10,000/\hat{\theta}$; $(5000 + 6000 + 10,000)/100,000 = 10,000/\hat{\theta}$

$21,000/100,000 = 10,000/\hat{\theta}$: Thus, $\hat{\theta} = 10,000/.21 =$ 47,619 cycles (ans.

GENERAL TWO-LEVEL DUTY CYCLE WITH DIFFERENT WEIBULL SLOPES

		Weibull Parameters
One Round	Stress S_1 : N_1 cycles	b_1 and θ_1
	Stress S_2 : N_2 cycles	b_2 and θ_2

$$\mathcal{E}_1(N_1) = (N_1/\theta_1)^{b_1} = (N_2'/\theta_2)^{b_2}, \quad \text{or} \quad N_2' = \theta_2(N_1/\theta_1)^{b_1/b_2}$$

$$\mathcal{E}_2\left[\theta_2(N_1/\theta_1)^{b_1/b_2} + N_2\right] = \left[(N_1/\theta_1)^{b_1/b_2} + N_2/\theta_2\right]^{b_2} = (N_1''/\theta_1)^{b_1}$$

$$\therefore \quad N_1'' = \theta_1\left[(N_1/\theta_1)^{b_1/b_2} + N_2/\theta_2\right]^{b_2/b_1}$$

$$\mathcal{E}_1\left\{\theta_1\left[(N_1/\theta_1)^{b_1/b_2} + N_2/\theta_2\right]^{b_2/b_1} + N_1\right\} = \left\{\left[(N_1/\theta_1)^{b_1/b_2} + N_2/\theta_2\right]^{b_2/b_1} + N_1/\theta_1\right\}^{b_1}$$

$$= (N_2''/\theta_2)^{b_2}$$

$$\therefore \quad N_2'' = \left\{\left[(N_1/\theta_1)^{b_1/b_2} + N_2/\theta_2\right]^{b_2/b_1} + N_1/\theta_1\right\}^{b_1/b_2}$$

$$\mathcal{E}_2(2 \text{ Rounds}) = \left(\left\{\left[(N_1/\theta_1)^{b_1/b_2} + N_2/\theta_2\right]^{b_2/b_1} + N_1/\theta_1\right\}^{b_1/b_2} + N_2/\theta_2\right)^{b_2}$$

$$\mathcal{E}_1(2\tfrac{1}{2} \text{ Rounds}) = \left(\left(\left\{\left[(N_1/\theta_1)^{b_1/b_2} + N_2/\theta_2\right]^{b_2/b_1} + N_1/\theta_1\right\}^{b_1/b_2} + N_2/\theta_2\right)^{b_2/b_1} + N_1/\theta_1\right)^{b_1}$$

$$\mathcal{E}_2(3 \text{ Rounds}) = \left(\left(\left(\left\{\left[(N_1/\theta_1)^{b_1/b_2} + N_2/\theta_2\right]^{b_2/b_1} + N_1/\theta_1\right\}^{b_1/b_2} + N_2/\theta_2\right)^{b_2/b_1} + N_1/\theta_1\right)^{b_1/b_2} + N_2/\theta_2\right)^{b_2}$$

The entire procedure for a two-level duty cycle with different Weibull slopes for the two stress levels can be summarized step by step , as follows :

Step 1 : Let $\lambda_1 = (N_1/\theta_1)$

Step 2 : Calculate $\lambda_1^{b_1/b_2}$

Step 3 : Add (N_2/θ_2) to get λ_2

Step 4 : Calculate $\lambda_2^{b_2/b_1}$

Step 5 : Add (N_1/θ_1) to get λ_3

Step 6 : Calculate $\lambda_3^{b_1/b_2}$

Step 7 : Add (N_2/θ_2) to get λ_4

Step 8 : Calculate $\lambda_4^{b_2/b_1}$

Step 9 : Add (N_1/θ_1) to get λ_5

Step 10: Calculate $\lambda_5^{b_1/b_2}$

Step 11 : Add (N_2/θ_2) to get λ_6

etc. etc.

Entropy Calculations at the End of the Rounds of the Duty Cycle

(End of Round 1)

(1 Round) $= \lambda_2^{b_2}$

(End of Round 2)

(2 Rounds) $= \lambda_4^{b2}$

(End of Round)

(3 Rounds) $= \lambda_6^{b_2}$

etc .

CONCLUSION

It can be seen that the Entropy approach constitutes a systematic method of evaluating cumulative damage produced by a given duty cycle with changing stress levels. It is a decided improvement over the old-fashioned techniques which were employed prior to the discovery of the Entropy Method. Such old-time rules as Miner's Rule and the Corten-Dolan Equation are simply the theoretical outcomes of special situations when analyzed by the Entropy Method.