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APPLICATIONS OF THE CONCEPTS OF STATISTICAL WEAKNESS AND STATISTICAL STRENGTH IN PRODUCT ASSURANCE

INTRODUCTION

It is amazing how much easier it can be to understand and apply certain statistical principles of data analysis if they can first be expressed in terms of some elementary definitions which precisely define the workings of the basic statistical laws in nature and the world of probabilistic predictions . Two concepts which always apply to the study of product reliability and assurance are

(I) The Concept of Statistical Weakness

(II) The Concept of Statistical Strength

From these two concepts have arisen a multitude of useful and practical approaches to such questions as

(a) The Significance of Reliability Growth

(b) The Vulnerability of a System of Components in Series

(c) The Durability of Parallel Redundancy

(d) The Confidence of Complying to a Reliability Goal

(e) Trade-Off Principles in Accelerated Testing

(f) Confidence Bands for Forecasting

In this bulletin we shall elaborate on all these applications after defining the concepts of Statistical Weakness and Statistical Strength .

THE CONCEPT OF STATISTICAL WEAKNESS

DEFINITION: If a population of items (such as systems or components) used in a certain product has a cumulative distribution of life x given by the function F(x), where

$F(x)$ = cumulative fraction of the population failed in service time x,

then at time x, the Statistical Weakness of the product is defined to be

$$\text{Statistical Weakness (at } x) = \mathcal{E}(x) = -\ln\left[1 - F(x)\right]$$

If $R(x) = 1 - F(x)$ = Reliability to x,

Then

$$\text{Statistical Weakness (at } x) = \mathcal{E}(x) = -\ln R(x) \quad *$$

PROPERTIES OF STATISTICAL WEAKNESS (OR ENTROPY)

The concept of Statistical Weakness (or Entropy) defined by

$$\mathcal{E}(x) = -\ln\left[1 - F(x)\right]$$

$F(x)$ = CDF of the life x has the following properties:

PROPERTY 1 : The Statistical Weakness (Entropy) is zero at $x = 0$, if the life distribution's minimum life is zero. In case the minimum life is $x = \alpha$, then $\mathcal{E}(\alpha) = 0$. This is because $1 - F(x) = 1$, and $-\ln 1 = 0$.

PROPERTY 2 : The Statistical Weakness is a monotone increasing positive number which becomes larger and larger as the service time x is increased.

PROPERTY 3 : At the maximum Life U, the Statistical Weakness is $\mathcal{E}(U) = \infty$. This is because $1 - F(U) = 0$ and $-\ln 0 = +\infty$.

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* It should be noted that this definition of Statistical Weakness is the very same definition as that of Entropy at x.

APPLICATION TO RELIABILITY GROWTH

When we are dealing with the question of reliability growth with respect to a target cumulative distribution function of life we can easily determine where we are with respect to the target by using the concept of Statistical Weakness (entropy) within the target cumulative distribution function $F_o(x)$. The technique is illustrated by the following example.

Suppose it has been established that the desired goal for durability is a Weibull population with Weibull slope b_o and characteristic life θ_o, i. e., the desired cumulative distribution function of life is

$$F_o(x) = 1 - e^{-\left(\frac{x}{\theta_o}\right)^{b_o}}$$

(x = life).

Suppose, furthermore, that N items are tested and run for the following time periods :

Item #1 is run for x_1 time units

Item #2 is run for x_2 time units

Item #3 is run for x_3 time units

. . . .

. . . .

. . . .

Item #N is run for x_N time units

Suppose that out of these N items, some number $r > 0$ are failed, while $(N - r)$ of them are unfailed at their running times.

The technique to be employed then is to calculate the <u>Average Entropy (Weakness) per Failure</u> within the target function $F_o(x)$.

Now, Statistical Weakness at $x = -\ln\left[1 - F_o(x)\right]$

Hence, for the data set $(x_1, x_2, x_3, \ldots x_N)$ of which <u>r</u> are failed and $(N - r)$ unfailed, we have

$$\text{TOTAL ENTROPY} = -\ln\left[1 - F_o(x_1)\right] - \ln\left[1 - F_o(x_2)\right] - \ldots - \ln\left[1 - F_o(x_N)\right]$$

and

AVERAGE ENTROPY PER FAILURE is

$$\mathcal{E}_{\text{Average}} = \frac{\text{Total Entropy}}{\text{No. of Failures}} = -\frac{\ln\left[1 - F_o(x_1)\right] + \ln\left[1 - F_o(x_2)\right] + \ldots + \ln\left[1 - F_o(x_N)\right]}{r}$$

$$\mathcal{E}_{\text{Average}} = \frac{(x_1/\theta_o)^{b_o} + (x_2/\theta_o)^{b_o} + \ldots + (x_N/\theta_o)^{b_o}}{r}$$

Then, the Confidence that the data set shows durability at least as good as the goal is the Normal area to a Z-score given by

$$Z = \sqrt{r}\,(\mathcal{E}_{\text{ave.}} - 1) .$$

This implies that any $\mathcal{E}_{\text{ave.}}$ below unity shows inferiority to the goal (conf. < 50%), while any $\mathcal{E}_{\text{ave.}}$ greater than unity shows superiority to the goal (conf. > 50%) .

APPLICATION TO A SYSTEM OF COMPONENTS IN SERIES

The case of an assembly with several components in series is illustrated in Figure 1 below :

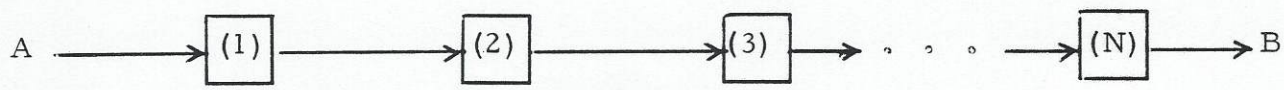

FIGURE 1

The system is represented by the path going from A to B . This path contains components (1), (2), (3), . . . (N) . All of the components must survive a target service time x if the system (A, B) is to survive for that same service time x. This Law of Components in Series making up a system can be expressed as follows :

ASSEMBLY STATISTICAL WEAKNESS (at x)

= SUM OF COMPONENTS STATISTICAL WEAKNESS to x .

This is expressed by the following equation :

$$\mathcal{E}_{\text{assembly}}(x) = \mathcal{E}_1(x) + \mathcal{E}_2(x) + \ldots + \mathcal{E}_N(x) \qquad (1)$$

or

$$-\ln R_{\text{assembly}}(x) = -\ln R_1(x) - \ln R_2(x) - \ldots - \ln R_N(x) \qquad (2)$$

or

$$R_{\text{assembly}}(x) = R_1(x) \cdot R_2(x) \cdots R_N(x) \qquad (3)$$

It can be seen that equation (3) is simply the well known Product Rule for Independent Components in Series . We can also state this law as follows :

Assembly Entropy (at x) = Sum of Component Entropies at x

THE CONCEPT OF STATISTICAL STRENGTH

DEFINITION : If a population of items (such as systems or components) used in a certain product has a cumulative distribution of life x given by the function F(x) , where

F(x) = Cumulative Fraction of the Population Failed in Service Time x ,

then at time x , the Statistical Strength of the product is defined to be

$$\text{STATISTICAL STRENGTH (at } x) = S(x) = -\ln F(x) \quad (4)$$

PROPERTIES OF STATISTICAL STRENGTH

The concept of Statistical Strength as defined above has the following properties:

PROPERTY 1 : The Statistical Strength at the minimum life (beginning) of a life distribution is infinite .

PROPERTY 2 : The Statistical Strength at the end of a life distribution (i. e., at the maximum life) is zero .

PROPERTY 3 : The Statistical Strength is a monotone decreasing positive number which becomes smaller and smaller as the service time x increases .

APPLICATION TO A SYSTEM OF COMPONENTS IN PARALLEL

An assembly of N components in parallel is schematically represented as shown in Figure 2 .

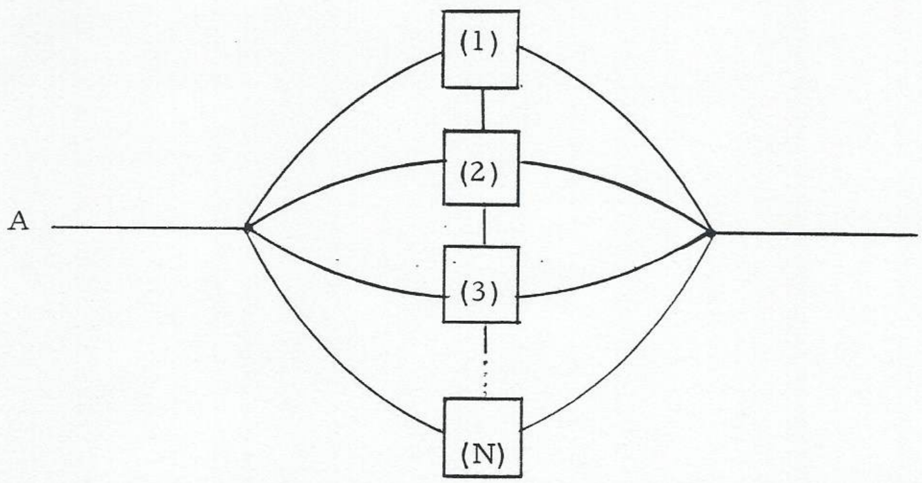

FIGURE 2

The path AB is unbroken as long as at least one of the N components, labeled (1), (2), (3), ... (N) , is unfailed. This implies that the probability of the system failing in time x is the product of all the failure probabilities of the individual components to that same time x. Mathematically, this is the product rule for cumulative distribution functions of failure of items in parallel . This product rule can be written as follows :

$$F_{assembly}(x) = F_1(x)\, F_2(x)\, F_3(x) \cdots F_N(x) \quad , \qquad (5)$$

where

$F_{assembly}(x)$ = Cumulative Distribution Function of Failures for the entire assembly

$F_i(x)$ = Cumulative Distribution Function of Failures for component #(i)

Taking natural logarithms of both sides of equation (5) :

$$\ln F_{assembly}(x) = \ln F_1(x) + \ln F_2(x) + \cdots + \ln F_N(x) \qquad (6)$$

Changing all the signs :

$$-\ln F_{assembly}(x) = -\ln F_1(x) - \ln F_2(x) - \ldots - \ln F_N(x) \qquad (7)$$

but , by the definition of Statistical Strength given in (4) , we can write (7) as follows

$$S_{assembly}(x) = S_1(x) + S_2(x) + \cdots + S_N(x) \qquad (8)$$

Equation (8) tells us that an assembly of components in parallel has a Statistical Strength equal to the Sum of the Statistical Strengths of all the components.

APPLICATIONS TO ACCELERATED TESTING

One important application of the concept of Statistical Weakness (Entropy) is in the analysis of accelerated testing. The most common method of shortening a test is to increase the stress level. Suppose, for example, that a standard stress yields a life distribution with Weibull parameters-- b_o = Standard Weibull Slope (at std. stress σ_o), θ_o = std. characteristic life (at standard stress σ_o). Now suppose, furthermore, that the test item has run x_o cycles at this initial stress σ_o and is still unfailed. In order to shorten the waiting time to failure we switch to a higher stress σ_1, for which the Weibull life parameters are b_1 = New Weibull Slope (at new stress σ_1) and θ_1 = New characteristic life (at new stress σ_1).

Now we run the specimen at the new stress σ_1 for x_1 more cycles.

QUESTION: What is the accelerated statistical weakness after x_o cycles at stress σ_o followed by x_1 cycles at stress σ_1?

ANALYSIS BY STATISTICAL WEAKNESS(ENTROPY)

If the specimen runs x_o cycles at stress σ_o, with Weibull parameters (b_o, θ_o), then the Entropy accumulated in x_o cycles at stress σ_o is

$$\mathcal{E}_o(x_o) = (x_o/\theta_o)^{b_o} .$$

At the second stress σ_1, with Weibull parameters (b_1, θ_1), the number of cycles x' which would produce the same accumulated Entropy $\mathcal{E}_o(x_o)$ is found by solving the following equation for x':

$$\left(\frac{x'}{\theta_1}\right)^{b_1} = \left(\frac{x_o}{\theta_o}\right)^{b_o}$$

This yields
$$x' = \theta_1 \left(\frac{x_o}{\theta_o}\right)^{(b_o/b_1)} .$$

Now , if we run the specimen for another x_1 cycles at stress σ_1 , the total number of cycles at the single stress σ_1 will total up to $(x' + x_1)$, i. e., in $(x' + x_1)$ cycles at stress σ_1 the Entropy accumulated is equivalent to that accumulated by x_o cycles at σ_o plus x_1 cycles at σ_1 . Hence, the accumulated Entropy after the second stress level operates for x_1 cycles is

$$\mathcal{E}_1 = \left(\frac{x' + x_1}{\theta_1} \right)^{b_1} \qquad \text{(Ans.)}$$

CONCLUSION

It can be seen what useful tools for the analysis of durability and reliability problems we have in the concepts of Statistical Weakness and Statistical Strength .