Statistical Bulletin
Reliability & Variation Research

DETROIT RESEARCH INSTITUTE
P.O. BOX 36504 • GROSSE POINTE, MICHIGAN 48236 • (313) 886-7976

LEONARD G. JOHNSON
EDITOR

WANG H. YEE
DIRECTOR

Volume 14
Bulletin 1

April , 1984
Page 1

THE μ TECHNIQUE OF CONSTRUCTING LOG-PARAMETRIC CONFIDENCE BANDS FOR CDF PLOTS OF FAILURE DATA

INTRODUCTION

As reliability analysts become more and more knowledgeable about the general field of reliability analysis they come to realize the terrible shortcomings of the classical non-parametric approach in dealing with the confidence levels involved in reliability estimation. This is all due to the fact that non-parametric confidence bands for plots of cumulative distributions as derived from order statistics are so shockingly wide as to be entirely useless for small sample sizes. In other words, non-parametric rank tables (say, for 5% and 95% confidence levels) are based on an assumption of a complete lack of prior knowledge about the underlying distribution function of a population whose statistical behavior we are attempting to predict.

In real life we aren't all that ignorant about all the properties of an underlying population distribution function, and, consequently, it makes much more sense to use whatever prior knowledge we might have about a life distribution. For example, in a good many situations we do know what the Weibull slope is for a given failure process, and in such situations we can legitimately use the so-called Log-Parametric Ranking of Order Statistics at confidence levels different from the Median (i.e., 50% confidence) level.

The purpose of this bulletin is to demonstrate the use of such a Log - Parametric Technique in constructing confidence bounds around a Median rank Weibull Plot (or any other type of cumulative plot, such as a Cumulative Entropy Plot), and to illustrate its tremendous practical uses and advantages due to its simplicity and the reasonable results which it yields for reliability predictions.

LOG-PARAMETRIC RELATIONS IN RELIABILITY

The typical situation in reliability analysis can be summarized as follows :

FIRST , We determine by som sort of special technique (such as Weibull plotting with median ranks) what the reliability is to any target life x with 50% confidence. This reliability with 50% confidence to target x is denoted by the symbol

$$R_{.50}(x) \quad .$$

Where the letter R denotes Reliability , and the subscript .50 denotes 50% confidence (i. e. , a median value of reliability), and x denotes the target life at which we are predicting the reliability (i. e. , the fraction of the population able to survive to life x).

SECOND , If we desire a confidence level different from 50% (say , confidence c) , we then are asking for the numerical value of the quantity denoted by the symbol

$$R_c(x) \quad ,$$

where the subscript has been changed from .50 to c . Thus if we want the predicted reliability to target life x with 95% confidence, we are asking for a number $R_{.95}(x)$, such that this number tells us that at least the fraction $R_{.95}(x)$ of the population is able to survive to life x with 95% confidence that the predicted fraction will not be any smaller , i. e., 95% confidence that the actual population reliability to life x will be at least as large as the predicted fraction .

For confidence $c > .50$, the reliability $R_c(x)$ promised to target x will obviously be less than $R_{.50}(x)$, because of the simple fact that demanding increased confidence in a minimum promise forces us to promise less (i. e. , a smaller reliability).

On the other hand, reducing the confidence c to a level less than 50% will allow the amount of reliability promised to be larger than $R_{.50}(x)$, i. e. , for $c < .50$, we have

$$R_c(x) > R_{.50}(x)$$

In any case , we can assume a general relation

$$R_c(x) = \left[R_{.50}(x)\right]^{\mu}, \quad \text{where} \tag{1}$$

is a positive exponent , with the requirement that when

$c > .50$ we must have $\mu > 1$, and when
$c < .50$ we must have $\mu < 1$.

- -

The Log-Parametric Formula for μ is

$$\mu = \left(\frac{c}{1 - c}\right)^{\frac{.55}{\sqrt{N}}} \tag{2}$$

Where

c = confidence level

N = sample size

$.55 = \frac{\sqrt{3}}{\pi}$ (A well known constant used in the Logistic function approximating the area under a Normal Density function)

INTERPRETING THE μ FACTOR IN TERMS OF ENTROPIES

Let us start with the relation (1), i.e.,

$$R_c(x) = \left[R_{.50}(x)\right]^{\mu} \tag{1}$$

Taking the natural logarithm of both sides of (1) :

$$\ln R_c(x) = \mu\left[\ln R_{.50}(x)\right] \tag{2}$$

<u>Changing Sign :</u>

$$-\ln R_c(x) = \mu\left[-\ln R_{.50}(x)\right] \tag{3}$$

but, $-\ln R_c(x)$ = Entropy at x with confidence c $= \mathcal{E}_c(x)$

and, $-\ln R_{.50}(x)$ = Entropy at x with 50% confidence $= \mathcal{E}_{.50}(x)$.

Thus, (3) becomes

$$\mathcal{E}_c(x) = \mu\, \mathcal{E}_{.50}(x) \tag{4}$$

Thus,

$$\mu = \frac{\mathcal{E}_c(x)}{\mathcal{E}_{.50}(x)} \tag{5}$$

In other words, the μ factor is the ratio of two entropies (at confidence c and at confidence .50) thus, to construct the <u>c</u> confidence boundary at point x in a cumulative failure fraction plot or in a cumulative entropy plot, we simply multiply the median entropy at x by the μ factor to obtain the entropy with confidence c, implying that the entropy at x will be at most $\mu\, \mathcal{E}_{.50}(x) = \mathcal{E}_c(x)$ with confidence c,

where

$$\mu = \left(\frac{c}{1-c}\right)^{\frac{.55}{\sqrt{N}}},$$

where N = sample size at life x .

ILLUSTRATING A TYPICAL SITUATION

Suppose N failure data points have been plotted on Weibull paper by using median ranks , thus yielding the graph shown in Figure 1 below :

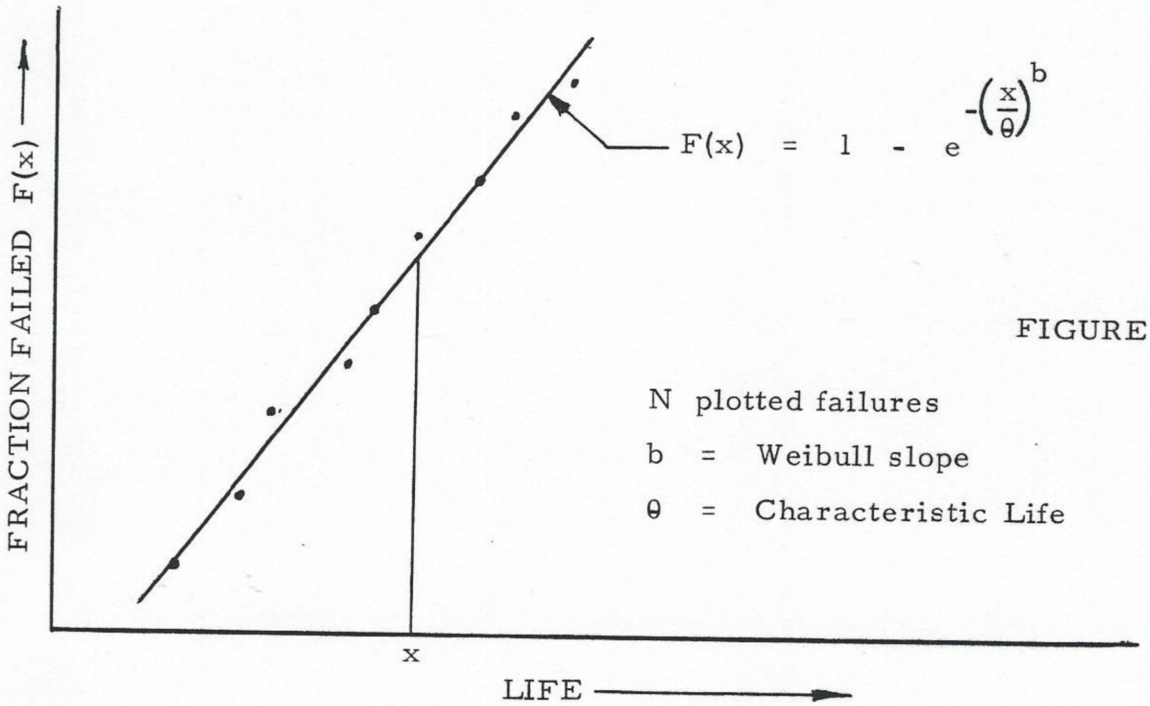

FIGURE 1

Taking any target x on the life axis , we have (from the median rank plot):

$$\mathcal{E}_{.50}(x) = \ln \frac{1}{1 - F(x)} = \left(\frac{x}{\theta}\right)^b .$$

If , now, we desire the upper 95% confidence boundary at life x , we must multiply $\mathcal{E}_{.50}(x)$ by the factor

$$\mu = \left[\frac{.95}{1 - .95}\right]^{\frac{.55}{\sqrt{N}}} = 19^{\frac{.55}{\sqrt{N}}}$$

if the sample size is N = 5 (for example) , then

$$\mu = 19^{.55/\sqrt{5}} = 2.06315 .$$

Using this μ factor , we calculate $\mathcal{E}_{.95}(x) = \mu \, \mathcal{E}_{.50}(x)$,

i.e. , $\mathcal{E}_{.95}(x) = 2.06315 \, \mathcal{E}_{.50}(x)$.

For example , suppose we have the Weibull parameters

$$b = 2.5 \qquad \text{(Weibull slope)}$$

$$\theta = 1000 \text{ hours (Characteristic Life)}$$

At characteristic life : $\mathcal{E}_{.50}$ (1000 hours) = 1 (always unity at $x = \theta$)

then $\mathcal{E}_{.95}$ (1000 hours) $= \mu \, \mathcal{E}_{.50}$ (1000 hours)

$$= 2.06315\,(1) = 2.06315$$

thus, $F_{.95}\text{(1000 hrs.)} = 1 - e^{-\mathcal{E}_{.95}\text{(1000 hrs.)}}$

$$= 1 - e^{-2.06315} = .873 \quad .$$

Thus the upper 95% bound at x = 1000 hrs., is (in this case) at the ordinate 87.3% on the cumulative Weibull vertical axis. In other words, the upper 95% confidence boundary looks as follows :

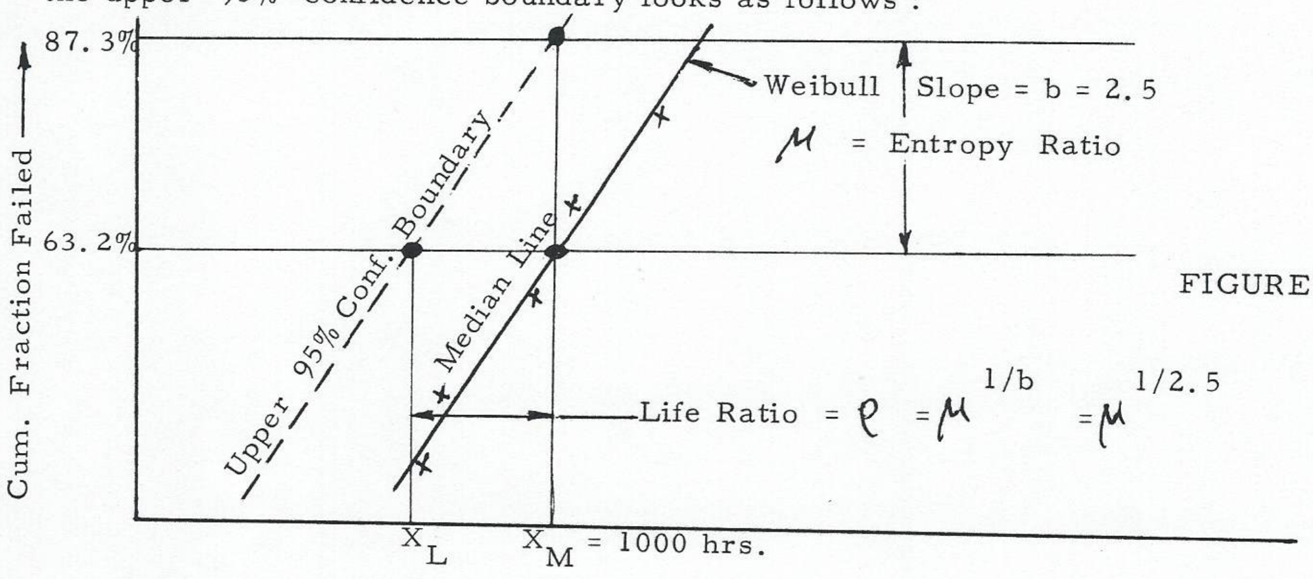

FIGURE 2

Thus , for a sample of 5 failures , the 95% confidence boundary (log-parametric) is parallel to the median line and passes through 87.3% at θ = 1000 hrs., and

$$\mu = \frac{\text{Entropy on 95\% Confidence Line}}{\text{Entropy on 50\% Confidence Line}} = 2.06315 \quad .$$

DETERMINING THE LIFE RATIO ρ AT ANY LEVEL

In Figure 2 we see that the 95% confidence boundary (dotted line) is to the left of the Median line by a fixed life ratio ρ at any quantile level, such that

$$\frac{X_M}{X_L} = \rho .$$

Furthermore, because the Weibull slope is b, it follows that

$$\rho^b = \mu$$

since $b \ln \rho = \ln \mu$, or $b = \dfrac{\ln \mu}{\ln \rho}$.

(by definition of slope on the log-log plot of Entropy vs. Life)

Thus,
$$b = \frac{\ln\left(\dfrac{\mathcal{E}_{.95}}{\mathcal{E}_{.50}}\right)}{\ln\left(\dfrac{X_M}{X_L}\right)}$$

Therefore, the ratio $(X_M / X_L) = \mu^{1/b} = (2.06315)^{1/2.5} = 1.336$.

$$\therefore \quad X_L = X_M / 1.336 = 1000/1.336 = 748.5 \text{ Hours.}$$

Thus, in this example, we can promise that the true population characteristic life is at least 748.5 hours with 95% confidence.

CONCLUDING REMARKS

The μ factor is a very handy tool for constructing a confidence boundary for a median rank plot, since it denotes the Entropy ratio between $\mathcal{E}_c(x)$ and $\mathcal{E}_{.50}(x)$, i.e.,

$$\mu = \frac{\mathcal{E}_c(x)}{\mathcal{E}_{.50}(x)}$$

Furthermore, at any quantile Q, the B_Q life ratio $\rho = \mu^{1/b} = \dfrac{.50\, B_Q}{c\, B_Q}$.
(b = Weibull slope)

It should be remembered that at each level the correct sample size should be employed by deleting all suspended items prior to that quantile level from the original total number of items we started out with at time x = 0. In other words, in the formula

$$\mu = (c/1 - c)^{.55\sqrt{N}}$$

, we should make sure we are using the correct value for the sample size N at the level which we are analyzing for the factors μ and $\rho = \mu^{1/b}$.