Statistical Bulletin
Reliability & Variation Research

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Volume 12 Bulletin 3 July , 1982

RELIABILITY ACCUMULATION THEORY

Reliability Accumulation Theory is a mathematical theory which enables us to sum up (i.e., accumulate) all the running times of similar items into a resultant success total T for the items to any required target. The theory is applicable to a collection of both failed and suspended (unfailed) items, regardless of their running times. The only requirement is that the Weibull slope and minimum life be known (or estimable). From T, the resultant success total to the required target (with zero failures), we can estimate the reliability to the required target with confidence C by means of the formula (X_o = Required Target)

$$R_C(X_o) = 1 - \text{C-rank of } 1^{st} \text{ in } (T+1)$$

$$= \underline{(1 - C)^{\frac{1}{T+1}}}$$

N_i items run to time X_i (different than the required target), and either unfailed or just failed at X_i are equivalent to

(A) $$N_o = N_i \left(\frac{X_i}{X_o}\right)^b$$ items run successfully to target X_o.

(NOTE: b = Weibull slope, and Minimum Life assumed to be zero)

The relationship given in (A) is derived as follows:

(A) PROB. (N_i successes in N_i trials to target X_i) = $\left[R(X_i)\right]^{N_i}$

(B) PROB. (N_o successes in N_o trials to target X_o) = $\left[R(X_o)\right]^{N_o}$

(A) and (b) must be equal probabilities if N_o successes to X_o are to be equivalent to N_i successes to X_i .

Hence,

$$\left[R(X_o)\right]^{N_o} = \left[R(X_i)\right]^{N_i}$$

or

$$R(X_o) = \left[R(X_i)\right]^{N_i/N_o}$$

Furthermore, we are given that $R(X_0) = EXP(-(X_o/\theta)^b)$

and $R(X_i) = EXP(-(X_i/\theta)^b)$

Hence,

$$R(X_o) = \left[R(X_i)\right]^{\left(\frac{X_o}{X_i}\right)^b}$$

$$\therefore \quad \frac{N_i}{N_o} = \left(\frac{X_o}{X_i}\right)^b$$

or

$$N_o = N_i\left(\frac{X_o}{X_i}\right)^b$$

Q. E. D.

Thus,

$$\left\{\begin{array}{l} N_1 \text{ items running to } X_1 \\ N_2 \text{ items running to } X_2 \\ \cdot \qquad\qquad\qquad \cdot \\ \cdot \qquad\qquad\qquad \cdot \\ \cdot \qquad\qquad\qquad \cdot \\ N_k \text{ items running to } X_k \end{array}\right\}$$

are, as a total data collection, equivalent to N_o items running to target X_0 (with zero failures) , where

$$N_o = N_1\left(\frac{X_1}{X_o}\right)^b + N_2\left(\frac{X_2}{X_o}\right)^b + \cdots + N_k\left(\frac{X_k}{X_o}\right)^b$$

b = WEIBULL SLOPE

X_o = REQUIRED TARGET

For each X_i , the number of items running to X_i is N_i .

The minimum life in the above analysis is ZERO .

A NUMERICAL EXAMPLE :

Suppose the Weibull slope is 2.

Suppose 10 items run 100 hours each

20 items run 50 hours each

5 items run 200 hours each

QUESTION: How many items would this amount to for a required target of 75 hours ?

SOLUTION

In this case : $N_1 = 10$ $X_1 = 100$

$N_2 = 20$ $X_2 = 50$

$N_3 = 5$ $X_3 = 200$

Furthermore, $X_0 = 75$ (Required Target)

$b = 2$ (Weibull slope) Min. Life = 0 .

Hence,

$$N_o = N_1\left(\frac{X_1}{X_o}\right)^b + N_2\left(\frac{X_2}{X_o}\right)^b + N_3\left(\frac{X_3}{X_o}\right)^b$$

$$= 10\,(100/75)^2 + 20\,(50/75)^2 + 5(200/75)^2$$

$$= \frac{350000}{5625} = 62$$ Successes to 75 hrs. (Ans.)

Entropy is defined (in a statistical sense) as the NATURAL LOGARITHM OF THE RECIPROCAL OF THE RELIABILITY.

Hence, the ENTROPY TO TARGET X_i is $\ln \frac{1}{R(X_i)} = \mathcal{E}(X_i)$

In case of two-parameter Weibull reliability :

$$R(X_i) = \mathrm{EXP}(-(X_i/\theta)^b)$$

and ENTROPY TO X_i $= \ln \frac{1}{R(X_i)} = \left(\frac{X_i}{\theta}\right)^b = \mathcal{E}(X_i)$

Taking the relation

$$N_o = N_1\left(\frac{X_1}{X_o}\right)^b + N_2\left(\frac{X_2}{X_o}\right)^b + \cdots + N_k\left(\frac{X_k}{X_o}\right)^b ,$$

and multplying each term by X_o^b gives

$$N_o X_o^b = N_1 X_1^b + N_2 X_2^b + \cdots + N_k X_k^b$$

Now, dividing each term by θ^b :

(B)
$$N_o\left(\frac{X_o}{\theta}\right)^b = N_1\left(\frac{X_1}{\theta}\right)^b + N_2\left(\frac{X_2}{\theta}\right)^b + \cdots + N_k\left(\frac{X_k}{\theta}\right)^b$$

or $$N_o\,\mathcal{E}(X_o) = N_1\,\mathcal{E}(X_1) + N_2\,\mathcal{E}(X_2) + \cdots + N_k\,\mathcal{E}(X_k)$$

The last equation in (B) states that the equivalent probabilistic conditions exist whenever ENTROPIES add up to the same TOTAL. This is known as the ENTROPY EQUIVALENCE THEOREM .

ILLUSTRATIVE EXAMPLE USING ENTROPIES

Suppose the Weibull slope is 3.

Suppose we run 100 items for 100 hours each, and 200 items for 25 hours each (all without failures) .

QUESTION: How many items N_o could we run for 50 hours each (without failure) with an equivalent probability for the latter accomplishment ?

SOLUTION

$$N_o \mathcal{E}(50) = 100 \mathcal{E}(100) + 200 \mathcal{E}(25)$$

$$\mathcal{E}(50) = \left(\frac{50}{\theta}\right)^3 \; ; \quad \mathcal{E}(100) = \left(\frac{100}{\theta}\right)^3 \; ; \quad \mathcal{E}(25) = \left(\frac{25}{\theta}\right)^3$$

Thus,

$$N_o \left(\frac{50}{\theta}\right)^3 = 100 \left(\frac{100}{\theta}\right)^3 + 200 \left(\frac{25}{\theta}\right)^3$$

or,

$$125000 \, N_o = 100{,}000{,}000 + 3{,}125{,}000 = 103{,}125{,}000$$

Thus,

$$N_o = \frac{103{,}125{,}000}{125{,}000} = 825 \text{ Successes to 50 hrs. each}$$

(ANS.)

The ENTROPY SUMMATION EQUATION at the end of (B) is universally valid for all types of distributions. All we need to do is to correctly define the ENTROPY FUNCTION $\mathcal{E}(X)$ in accordance with the distribution we have.

For example, if the distribution is a 3-parameter Weibull

$$F(X) = 1 - \text{EXP}\left[-\left(\frac{X-\alpha}{\theta-\alpha}\right)^{b}\right]$$

where α = MINIMUM LIFE

θ = CHARACTERISTIC LIFE

b = WEIBULL SLOPE

Then, the ENTROPY FUNCTION $\mathcal{E}(X)$ is

$$\mathcal{E}(X) = \left(\frac{X-\alpha}{\theta-\alpha}\right)^{b} .$$

Equal entropy sums imply equal PROBABILITIES of accomplishment.

Another name for the probability of accomplishment is COLLECTIVE RELIABILITY .

ILLUSTRATION WITH A 3-PARAMETER WEIBULL

Suppose $b = 2$ (Weibull slope)

$\alpha = 100$ Hrs. (Minimum Life)

$\theta = 1000$ Hrs. (Characteristic Life)

QUESTION: How many successes N_o to 200 hours have the same probability of accomplishment as 50 successes to 150 Hrs. each plus 20 successes to 300 Hrs. each ?

SOLUTION

$$N_o \, \mathcal{E}(200) = 50 \, \mathcal{E}(150) + 20 \, \mathcal{E}(300)$$

or $$N_o \left(\frac{200 - 100}{1000 - 100}\right)^2 = 50 \left(\frac{150 - 100}{1000 - 100}\right)^2 + 20 \left(\frac{300 - 100}{1000 - 100}\right)^2$$

or $$N_o \, (100)^2 = 50 \, (50)^2 + 20 \, (200)^2$$

or $$10000 \, N_o = 125000 + 800000 = 925000$$

Thus, $$N_o = \frac{925000}{10000} = 92$$ SUCCESSES to 200 Hrs. Each (ANS.)