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# BAYESIAN ANALYSIS OF RELIABILITY

Using a prior distribution with a minimum reliability at A and a modal reliability at unity when N consecutive sucesses are observed in a test.

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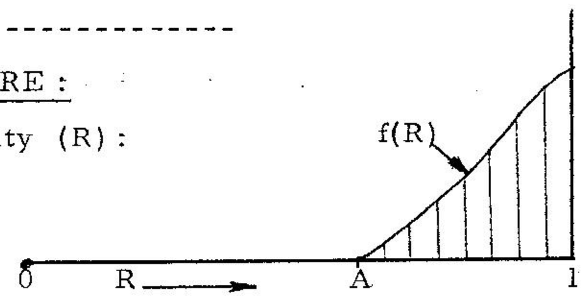
## STEP I IN THE BAYESIAN PROCEDURE :

Assume a prior PDF for the Reliability (R) :




$$f(R) = K(1 - A/R)^N$$

where K is such that $\int_A^1 K(1 - A/R)^N \, dR = 1$

Thus, $$\boxed{\text{Prob.}(R \leq \text{Rel.} \leq R + dR) = K(1 - A/R)^N \, dR} \quad (1)$$

## STEP II IN THE BAYESIAN PROCEDURE :

Take the data ( N sucesses in N trials) and write its conditional probability.

Thus, $$\boxed{\text{Prob.}(\text{Data If } R \leq \text{Rel.} \leq R + dR) = R^N} \quad (2)$$

## STEP III IN THE BAYESIAN PROCEDURE :

Multiply (1) and (2) to get the joint probability :

Thus,

$$\boxed{\begin{aligned}&\text{Prob.}(\text{Data and } R \leq \text{Rel.} \leq R + dR)\\ &= KR^N(1 - A/R)^N \, dR = K(R - A)^N \, dR\end{aligned}}$$

STEP IV IN THE BAYESIAN PROCEDURE :

Use Bayes' Theorem to obtain the posterior probability , i.e. Prob.(Hyp. if Data) . In this case ,

Hyp. is ( $R \leq$ Rel. $\leq R + dR$)

Data is ( N Sucesses in N Trials)

According to Bayes' Theorem :

$$\text{Prob.( Hyp. if Data )} = \frac{\text{Prob.( Data and Hyp.)}}{\text{Prob.(Data)}}$$

In this case ,

$$\text{Prob.(Data and Hyp.)} = \text{Prob.(Data and } R \leq \text{Rel.} \leq R + dR)$$

$$= K(R - A)^N dR$$

$$\text{Prob.(Data)} = \int_A^1 K(R - A)^N dR = \left[\frac{K(R-A)^N}{N+1}\right]_A^1 = \frac{K(1-A)^{N+1}}{N+1}$$

∴ The Posterior Probability of the Hypothesis is

$$\text{Prob.}\left[(R \leq \text{Rel.} \leq R + dR) \text{ if Data}\right]$$

$$= \frac{K(R-A)^N dR}{\dfrac{K(1-A)^{N+1}}{N+1}} = \frac{(N+1)(R-A)^N}{(1-A)^{N+1}} dR = g(R)\, dR$$

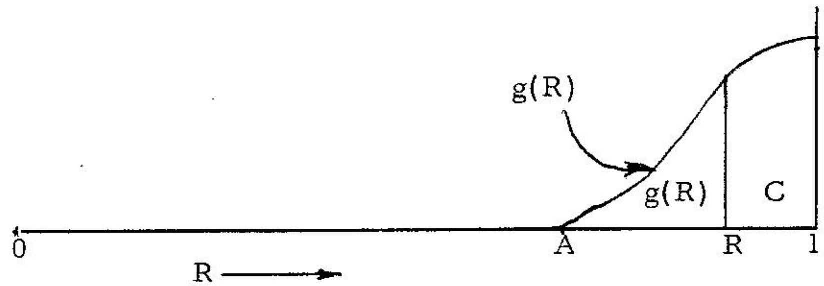
Thus , the Posterior PDF of R is

$$g(R) = \frac{(N+1)(R-A)^N}{(1-A)^{N+1}}$$

∴ The Posterior CDF of R is

$$G(R) = \int_A^R g(R)\, dR = \left[\frac{(R-A)^{N+1}}{(1-A)^{N+1}}\right]_A^R = \left(\frac{R-A}{1-A}\right)^{N+1}$$

We have the following picture of the Posterior Distribution of Reliability R :




The confidence that Rel. $\geq$ R is C , where

$$C = 1 - G(R)$$

$$\therefore \quad G(R) = 1 - C$$

$$\left[\frac{R - A}{1 - A}\right]^{N+1} = 1 - C$$

or $$C = 1 - \left[\frac{R - A}{1 - A}\right]^{N+1} \quad \text{or} \quad \frac{R - A}{1 - A} \times (1 - C)^{\frac{1}{N+1}}$$

or $$R = A + (1 - A)(1 - C)^{\frac{1}{N+1}}$$

Formula for Reliability having Confidence C when N successes are observed in N Trials , assuming a minimum Reliability of A.

SUCCESS RUN REQUIREMENT TABLE
A = 0 vs. A = .80 A = Minimum
(Worst Reliability) possible
(Confidence Desired = .90)

| RELIABILITY DESIRED | N (FOR A = 0) | N (FOR A = .80) |
|---|---|---|
| .90 | 21 | 3 |
| .95 | 44 | 8 |
| .99 | 229 | 44 |
| .999 | 2301 | 459 |
| .9999 | 23024 | 4604 |

## SAMPLE SIZE FORMULAS

( To Demonstrate a Desired $R_{.90}$)
(Confidence = 90%)

### ZERO DEFECTIVES

MINIMUM RELIABILITY = 0

$$N = \frac{2R_{.90} + .3026}{1 - R_{.90}}$$

MINIMUM RELIABILITY = .50

$$N = \frac{2R_{.90} - .8487}{1 - R_{.90}}$$

### 1 DEFECTIVE

MINIMUM RELIABILITY = 0

$$N = \frac{2R_{.90} + 1.888}{1 - R_{.90}}$$

MINIMUM RELIABILITY = .50

$$N = \frac{2R_{.90} - .056}{1 - R_{.90}}$$

### 2 DEFECTIVES

MINIMUM RELIABILITY = 0

$$N = \frac{2R_{.90} + 3.322}{1 - R_{.90}}$$

MINIMUM RELIABILITY = .50

$$N = \frac{2R_{.90} + .661}{1 - R_{.90}}$$

## SAMPLE SIZE TABLE FOR ZERO DEFECTIVES

| DESIRED $R_{.90}$ | N (for A = 0) | N (for A = .50) |
|---|---|---|
| .90 | 21 | 10 |
| .95 | 44 | 21 |
| .99 | 229 | 114 |
| .999 | 2301 | 1150 |
| .9999 | 23024 | 11511 |

## SAMPLE SIZE TABLE FOR 1 DEFECTIVE

| DESIRED $R_{.90}$ | N (for A = 0) | N (for A = .50) |
|---|---|---|
| .90 | 37 | 18 |
| .95 | 76 | 37 |
| .99 | 387 | 193 |
| .999 | 3886 | 1942 |
| .9999 | 38878 | 19438 |

## SAMPLE SIZE TABLE FOR 2 DEFECTIVES

| DESIRED $R_{.90}$ | N (for A = 0) | N (for A = .50) |
|---|---|---|
| .90 | 52 | 25 |
| .95 | 105 | 52 |
| .99 | 531 | 265 |
| .999 | 5320 | 2659 |
| .9999 | 53218 | 26608 |

## THEOREM

For D defectives in N trials the sample size (N) for a Minimum Reliability of A is equal to the sample size (N) for a minimum reliability zero multiplied by the factor ( 1 - A ) .

According to this, the sample size for A = .50 is half the sample size for A = 0 .

All this follows from a prior PDF of Reliability given by the formula

$$f(R) = K(1 - A/R)^{N-D}$$

Where K is such that

$$\int_A^1 K(1 - A/R)^{N-D} \, dR = 1 .$$